DIRECTIONAL RADIATION OF SOUND

IRVING WOLFF, *RCA Victor Co.*,
AND
LOUIS MALTER, *RCA Photophone*

INTRODUCTION

The chief factors which are a measure of the performance of loud speakers or combinations of loud speakers are:

1. Frequency range.
2. Unisformity of response.
3. Directional characteristics.
4. Efficiency.
5. Power handling capacity.

The greater part of the literature, which has been published in the English language on the subject of loud speakers, has been concerned with the frequency response or the efficiency. The directional characteristics have received very little attention as such, although a certain amount of the literature, which has been published on radiation from antennas[1,2] and on optical diffraction phenomena[3] may, with proper interpretation, be adopted to the acoustical effects which take place when sound is radiated from a loud speaker or a combination of loud speakers.

With this apology, we are, therefore, going to proceed to develop for acoustical purposes a certain amount of work which has already appeared in the above mentioned sources, as well as some new work, which is particularly applicable to the acoustical problems, so that a rather complete source for this subject may be made available to those interested. Additional derivations and sources, may be found in the various references.

In the following discussion of the directional characteristics of sound radiating systems, we shall proceed by first investigating the characteristics which a sound source should have in order to give the most satisfactory results under its conditions of use. As will be made clear below, the directional requirements for a loud speaker to be used in the

[1] Foster—Directive Diagrams of Antenna Arrays, Bell System Technical Journal, April 1926.

[2] Greshy—Richtcharacteristik von antennencombination, Zeitschrift fur Hochfrequenztechnik, Oct. and Nov. 1929.

[3] Drude—Theory of Optics. Chapters on Interference and Diffraction.

home are not the same as for one to be used in a theatre. The directional characteristics of various types and combinations of point, line and surface sources will be discussed in terms of the results obtained on a theoretical basis. The derivation of the results are carried thru for a number of typical cases in the appendix. The actual characteristics of various experimental and commercial sound sources will be compared and checked with the corresponding theoretical derivations, and will be discussed from a standpoint of the ideal characteristics which were proposed at the beginning of the discussion.

It is interesting to note that from a wave standpoint the radiation of sound at audible frequencies includes the same type of phenomena as diffraction in optics, and the radiation and reception of the electromagnetic waves used in communication and broadcasting. The results which are obtained in one field may be of either theoretical or experimental aid in any of the others. We may also point out that a considerable portion of this discussion may be of use in determining the directional pickup of sound, when receiving apparatus, having the same relative characteristics as that of the sound sources, which are discussed, is substituted for them.

Loud Speaker Directional Characteristic Requirements

For loud speakers to be used in small rooms, home use, a non-directional characteristic at all frequencies is to be desired. The listener is almost always close to the set, and sound absorption of objects in the room is usually high, so that the direct sound reaching the listener is of most importance.

Curves will be given below showing to what extent some of the loud speakers commonly used in radio sets satisfy these requirements.

Requirements for loud speakers which are used in auditoriums are more complex and difficult to satisfy. It is rather common opinion,[4, 5] that an auditorium in which speech is reproduced on loud speakers should have a lower reverberation period than one which is used for direct talking. This is due to the normally higher sound level of the mechanical reproduction, the large amount of power available, and the fact that a certain amount of reverberation has already been introduced in the recording. We are in agreement with Watson[6] that opti-

[4] S. K. Wolf, Proc. S.M.P.E.—14, 151, 1930.

[5] E. W. Kellogg—Proc. S.M.P.E.—14, 96, 1930.

[6] F. R. Watson—Science—67, 335, 1928.

mum conditions, at least for mechanical and electrical reproduction of speech, are obtained with equivalent outdoor conditions.

The reverberation time of an auditorium is usually defined in terms of an initial uniform or at least random distribution of sound energy. When the sound issues from loud speakers, it is probably fairer to measure an *effective reverberation time* in terms of an initial intensity, which is expressed in terms of the sound energy due to direct radiation from the loud speakers, plus the random sound energy, rather than in terms of an average uniform pressure throughout the auditorium. If the sound energy radiated from the loud speakers is directed in a beam towards the audience the effective reverberation is less than reverberation as usually defined; if it is directed away from the audience the effective reverberation time is greater. When the beam is not uniformly distributed, the audience in the stronger parts of the beam is conscious of the shortened effective reverberation. Therefore, if minimum effective reverberation is to be attained in any auditorium, we reach the conclusion that all sound energy should be directed so as to strike the audience before reflection from any other surface. From the standpoint of the effective efficiency of loud speakers to be used outdoors, this is also quite evidently the logical requirement.

If the sound radiation is taking place from a vibrating diaphragm or diaphragms, the directional characteristics are determined by the shape, size and relative positions of the diaphragms, the velocity distribution over their surfaces, and upon the distribution and configuration of whatever constraints there may be in the neighborhood of the vibrating surfaces such as baffles or enclosures. When sound is radiated from a horn or horns, the directional characteristics are determined almost entirely by the shape and size of the emerging waves and upon the intensity distribution over the horn mouth.

Theoretical Results

In considering the types of possible radiators we will proceed from the simple to the complex in the following order:

1. Point source.
2. Systems of point sources.
3. Line sources with uniform velocity distribution.
4. Line sources with non-uniform velocity distribution.
5. Curved line sources.
6. Plane surface sources—arbitrary distribution.

The consideration is very much simplified if the directional characteristics are determined at a point located at a sufficiently great distance from the sound radiating system so that lines joining the distant point with all parts of the sound radiating system are substantially parallel. Since the main facts of interest become apparent when this restriction is imposed, we will confine our attention to the characteristic at a great distance from the radiating system.

Curves have been plotted using either polar or rectangular co-ordinates, so as to be most useful for either illustration or computation. Most curves illustrating special cases and experimental curves have been plotted on polar paper, the radius being proportional to the absolute value of relative pressure in the direction of the angular co-ordinate. For general cases which are to be used for computation, it was found most convenient to plot on rectangular co-ordinates, using either the absolute or algebraic value of the relative pressure as ordinate, and a function of the angle of the point at which sound is received, frequency of radiation, and size of the source as abscissa.

1. *Point Source*

It is well known that the radiation from a point source is uniform in all directions. When a radiating source is small compared to the wave length being radiated, it behaves effectively like a point source.

2. *Systems of Point Sources*

Except for the particular case where the distance between the sources is less than or equal to one half wavelength the use of two point sources leads to an objectionable characteristic. Mathematically, (see Appendix, Section A for exact definition of R_α), the relative intensity on a circle in a plane passing thru the two points is:

$R_\alpha = \cos Z$ where $Z = (\pi d/\lambda) \sin \alpha$

d = distance between the two point sources

λ = wavelength

α = the angle the line from the source to the distant point makes with the normal to the line joining the two sources.

When the distance between the sources is less than one half wavelength, a symmetrical 180° displaced double maximum of gradually increasing sharpness is shown when the relative directional radiation is plotted on polar co-ordinate paper at increasing frequencies. At the point where the distance between the sources equals one-half wavelength these maxima become two lobes in contact. Above this frequency a number of other maxima begin to appear, rising to the same intensity as the central one, increasing in number as the frequency is raised.

We shall next consider a combination of point sources linearly arranged, intensities equal and all in phase (See Appendix, Section A, case (a)). Mathematically, the relative intensity in a circle in a plane passing through the line of points is given by:

$$R_\alpha = \frac{\sin nZ}{n \sin Z},$$

where n is equal to the number of point sources and the other symbols have the same meaning as above.

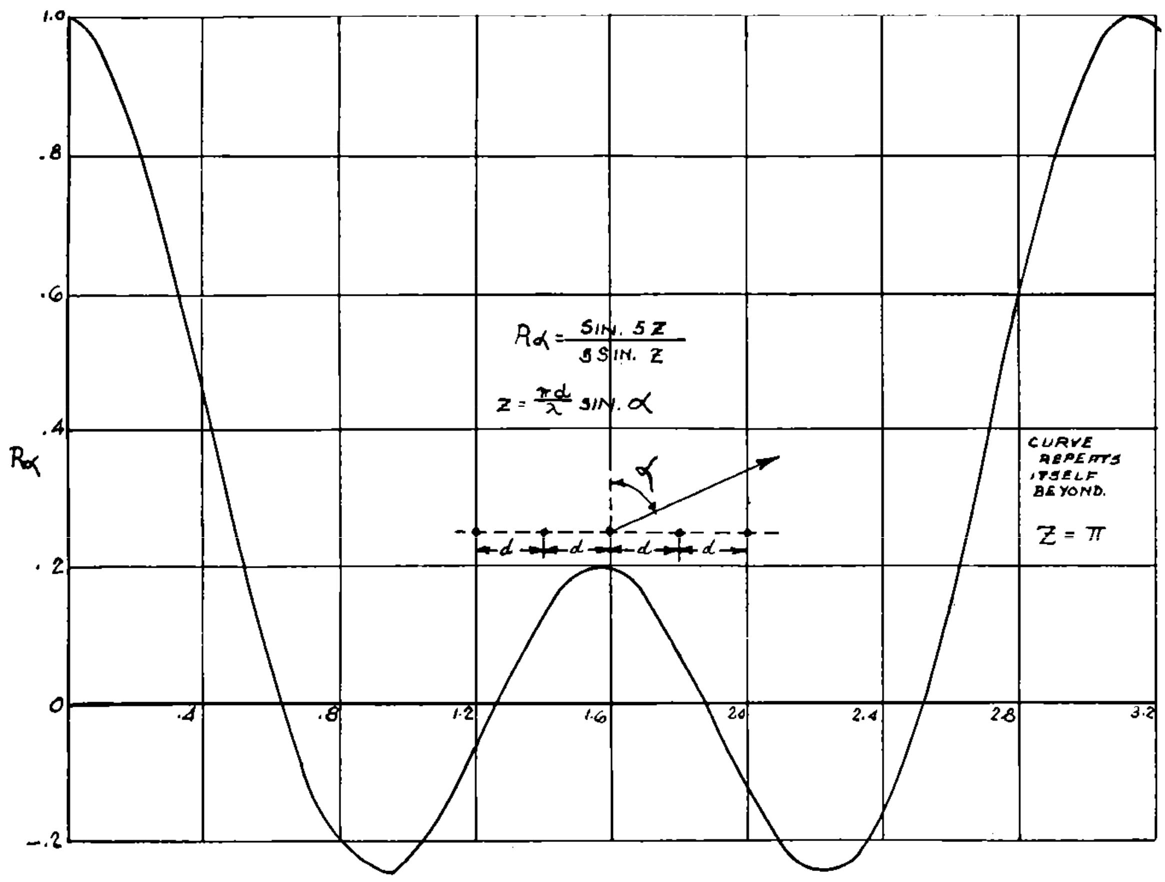

FIG. 1. *Theoretical Radiation Distribution Characteristic of Linear Arrangement of 5 Point Sources.*

The characteristic in this case is a function of n. As an illustration, the characteristic for $n=5$ has been plotted on Fig. (1) as a function of of Z, and in Figs. 2, 3 and 4 for some special cases. It will be noted that this characteristic consists of a large central maximum followed by three smaller ones of alternating phase and then another one which is of the same size as the central one. The curve then repeats itself in the

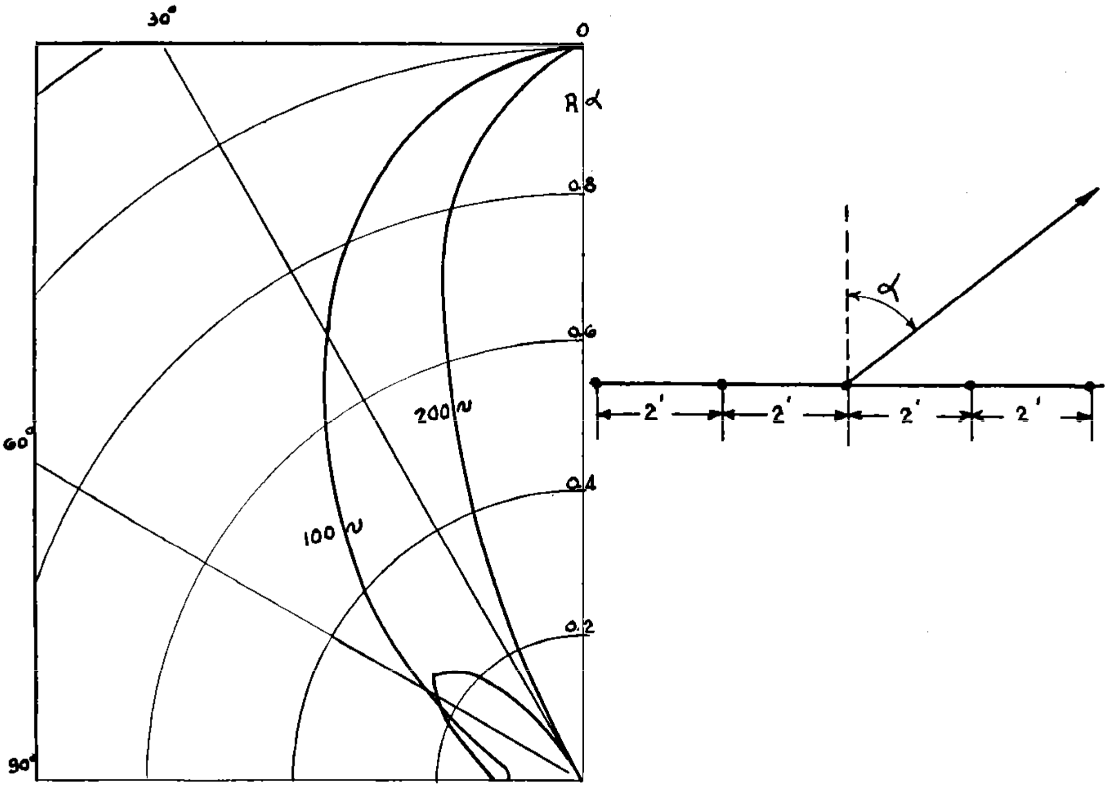

FIG. 2. *Theoretical Radiation Distribution Characteristics of 5 Point Sources on Line (Special Case).*

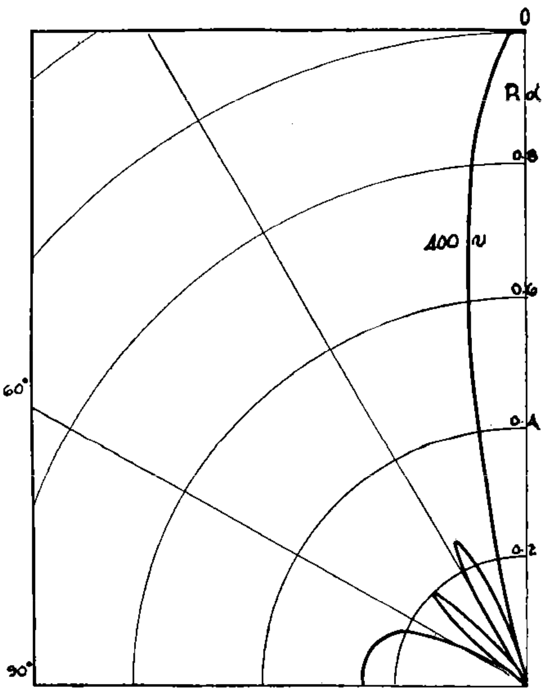

FIG. 3. *Theoretical Radiation Distribution Characteristic of 5 Point Sources on Line (Special Case).*

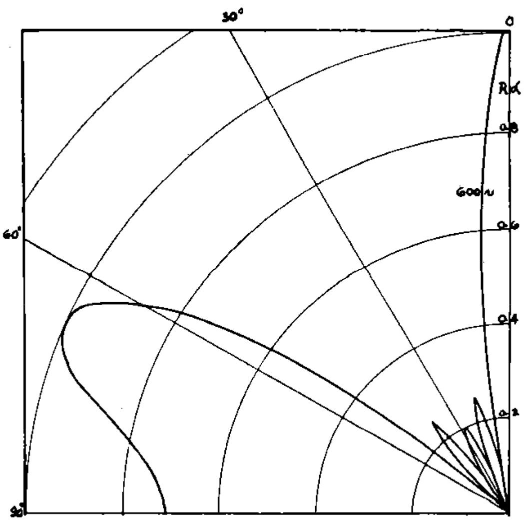

FIG. 4. *Theoretical Radiation Distribution Characteristic of 5 Point Sources on Line (Special Case).*

same manner. In general, the characteristic with n point sources will be quite similar to this with a number $n-2$ of small maxima between the large ones. At low frequencies there will first be present the single central maximum gradually increasing in sharpness as the frequency is raised and then this will be followed by the smaller maxima and later by the next large maximum. If the frequency range used is low enough so that the first large maximum which follows the central one does not appear, this directional characteristic is not very unsatisfactory. In a plane perpendicular to the line joining the source the distribution of sound intensity is, of course, uniform.

At first glance the negative values of the pressure may seem confusing. The negative values indicate a phase shift of 180° in sound pressure as the microphone is moved along a circle with the center of the source as center. An analysis of the computation of directional characteristics shows that for linear or plane sources which are symmetrical with respect to the center, only this 180° shift in phase is possible. In such a case, whenever the phase shifts, the intensity must pass thru zero. Unsymmetrical or non linear sources, on the other hand, have directional characteristics in which the phase assumes other values and, therefore, do not have directional characteristics in which the pressure must assume zero value between lobes.

It is interesting to consider the characteristic of a system similar to the one which has just been described with the exception that there is a progressive phase shift ϕ between successive point sources. Curves showing the characteristics of various sources of this type have been worked out by Stenzel[7,8] and Foster[1] (See Appendix, Section A, Case (b)). The result is expressed analytically in the form:

$$R_\alpha = \frac{\sin n\pi\left[\dfrac{d \sin \alpha}{\lambda} + \dfrac{\phi}{2\pi}\right]}{n \sin \pi\left[\dfrac{d \sin \alpha}{\lambda} + \dfrac{\phi}{2\pi}\right]}$$

This characteristic differs from that of the system of point sources with no phase shift, in that the direction of the principal maximum is no

[7] H. Stenzel—Uber die Richtcharacteristik von Schallstrahlern, Elektrische Nachrichten Technik, June 1927.

[8] H. Stenzel—Uber die Richtcharacteristik von in einer Ebene anegeordneten Strahlern, Elektrische Nachrichten Technik, May 1929.

Many other references may be obtained by referring to the bibliographies appended to the above papers.

longer necessarily perpendicular to the line joining the sources but has been shifted so that it is at an angle γ to the normal to the line of sources, where γ is $\sin^{-1}(\phi\lambda/2\pi d)$.

A radiating system of point sources may thus be made to radiate most strongly in any direction whatsoever by a proper phase shift being introduced for each source. However, the sharpness of the characteristic is a function of the angle of maximum radiation. In reference (8) the author describes a system in which an arrangement of an even number of point sources upon the circumference of a circle with proper phase distribution among the sources not only has the same "directional sharpness" (peilscharfe) in any direction, but has the added advantage of possessing small secondary maxima.

We have, up to this point, been considering a distribution of point sources on a straight line. We will now consider the effects which take place when the sources are distributed upon the arc of a circle. It will be assumed that the points are equally spaced and that the intensities and phases are equal.

The result obtained in this case (if the number of point sources is odd, i.e., if n is equal to $(2m+1)$, where m is an integer) is:

$$R_\alpha = \frac{1}{2m+1}\left| \sum_{k=-m}^{k=+m} \cos\left[\frac{2\pi R}{\lambda}\cos(\alpha+k\theta)\right] + i \sum_{k=-m}^{k=+m} \sin\left[\frac{2\pi R}{\lambda}\cos(\alpha+k\theta)\right] \right|$$

where θ is the angle subtended at the center of the circle by the arc joining any two successive point sources and α is the angle between a radius drawn thru the central point and the line joining the source and the distant point. The expression for an even number of sources is quite similar and need not be given here.

The directional distribution of radiation is a function of n, R/λ where R is the radius of the circle of which the arc is a part, and θ.

The characteristic for a source in which $n=(2m+1)=5$, $R=10$ feet and $\theta=7.5°$ at a frequency of 1400 cycles is plotted in Fig. 5. An inspection of the equation which has just been given shows that no change is made in R_α if θ is kept constant and R/λ is unchanged. For the purpose of comparison the characteristics of a similar linear arrangement is shown on the same figure.

It is seen that at this comparatively high frequency the characteristic of the circular combination although showing a tendency to exhibit the same peaks as the linear arrangement is considerably smoother. The characteristic in this case does not exhibit the same pronounced peaks as in the linear arrangement for the reason that except at very low frequencies it becomes impossible for the radiation from all the points to be in phase at a distant point in space. At lower frequencies the two curves will approach each other but, of course, large secondary maxima will not be present. Thus, this circular arrangement is on the whole more uniform than the similar linear arrangement. To facilitate comparison the absolute values of the relative pressure have been plotted for the linear arrangement.

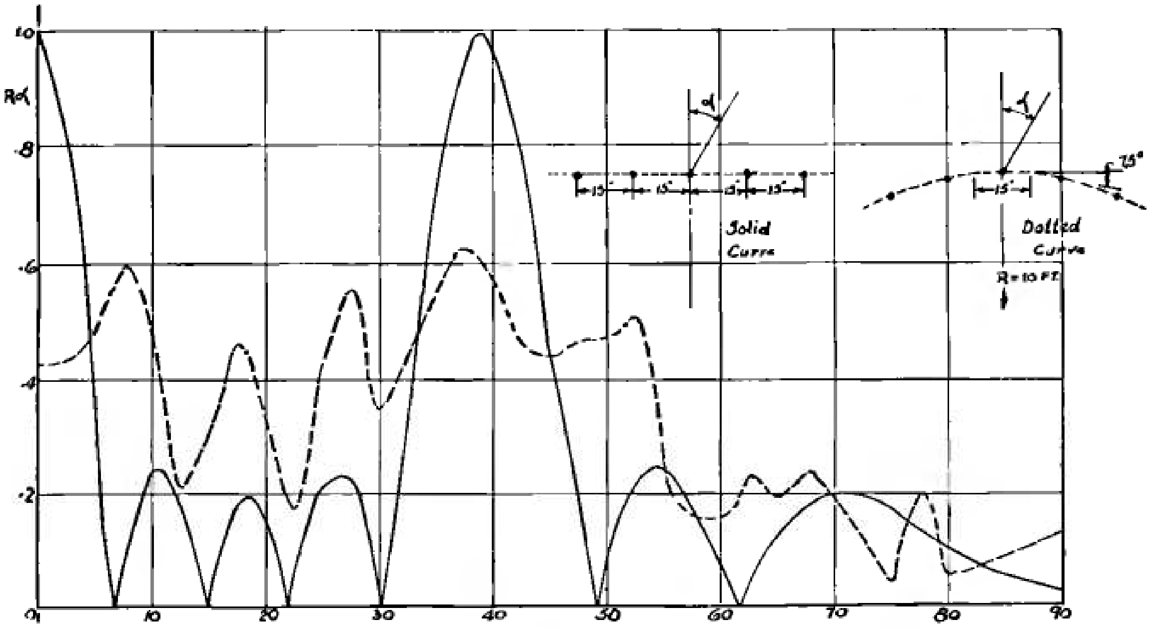

FIG. 5. *Theoretical Radiation Distribution Characteristics of Combination of Point Sources on Arc (Special Case at 1400 cycles).*

3. *Line Sources with Uniform Velocity Distribution*

We have seen that combinations of point sources whether on lines or sections of a circle lead to directional characteristics which contain an unsatisfactory number of secondary maxima. We shall now consider the results which can be obtained by means of continuous line sources. In this case the result obtained is:

$$R_\alpha = \frac{\sin Z}{Z}$$

where Z now equals $(\pi l/\lambda)$ sin α, and where l equals the length of the linear source (See Appendix, Section A, Case (b) for theory). The result has been plotted as a function of Z in Fig. 8 and for some special cases in Figs. 6 and 7.

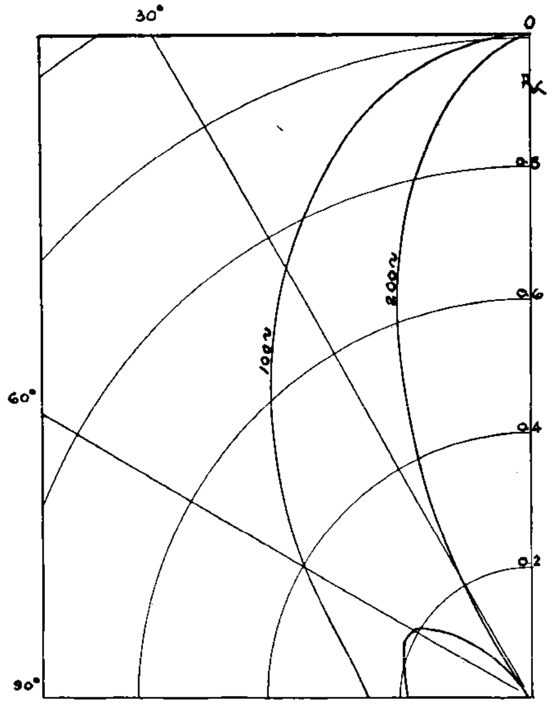

Fig. 6. *Theoretical Radiation Distribution Characteristic of Line Source (Special Case l= 9 ft.).*

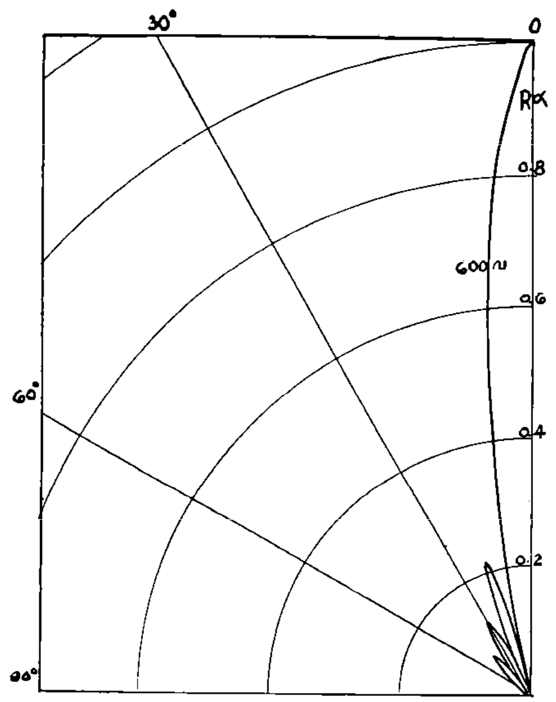

Fig. 7. *Theoretical Radiation Distribution Characteristic of Line Source (Special Case l=9 ft.).*

It will be seen that the secondary maxima of this radiator become progressively smaller, instead of rising to unity again as in the case in the combination of point sources. This is due to the fact that for directions other than the normal to the line all the radiation from all the points can never be in phase, whereas, in the sound source composed of point sources, secondary maxima equal in intensity to the radiation straight ahead appear at angles inclined to the normal when radiation from all points is in phase. In the case of the continuous line source the secondary maxima are much smaller, although not necessarily so small as not to exercise a disturbing influence.

The introduction of a progressive phase shift along a uniform line source introduces an effect similar to that noted with point sources (see Appendix, Section B, Case (b)).

4. *Line Sources with Non-uniform Velocity Distribution*

All the sources that have been studied up to this point have consisted of points which are all radiating with the same intensity or continuous sources in which the distribution of velocity is uniform. We shall now consider sources in which the distribution of velocity along the sources varies.

The study of a linear source with a distribution of velocity of the type $f(v)=e^{-mx}$ between $x=0$ and $x=l/2$ and where $f(v)=e^{mx}$ between $x=0$ and $x=-l/v$ where $f(v)$ represents the distribution in intensity along the source, m is a constant and l is the length of the source gives very interesting results. By changing the value of m, a wide variety of sources can be studied. When m is positive the intensity drops off from center to edge, when m is negative it increases, when m is zero, a source with uniform distribution of intensity is obtained. The larger m is, as a positive number, the more rapid the decrease in intensity towards the edges and as a limiting case for $m=\infty$ the single point source is obtained. On the other hand, the larger m is, as a negative number, the more rapidly the intensity increases towards the edges and as a limiting case for $m=-\infty$, 2 point sources are represented. We shall consider this distribution with all the points in phase. It can be shown that the relative sound intensity, at a distance, along a circle in the plane of the line for this exponential distribution is: (See Appendix, Section B, Case (c) for theory)

$$R_\alpha = \frac{\omega}{1-e^{-\omega}}\left[\frac{e^{-\omega}(Z\sin Z-\omega\cos Z)+\omega}{\omega^2+Z^2}\right]$$

In this equation $\omega=lm/2$ and $Z=(\pi l/\lambda)\sin\alpha$

It is evident that R_α is a function of both Z and ω. R_α as a function of Z for a series of values of ω has been plotted in Fig. 8. It will be noted that the value of R_α is a constant equal to one for $\omega=\infty$ as should be expected for a point source, and has a cosine form for $\omega=-\infty$ as should be expected for two point sources. As ω shifts from negative to positive values, the directional characteristics as functions of Z become less sharply defined and the secondary maxima become less intense. When $D=0$ the characteristic is that of the uniform line source.

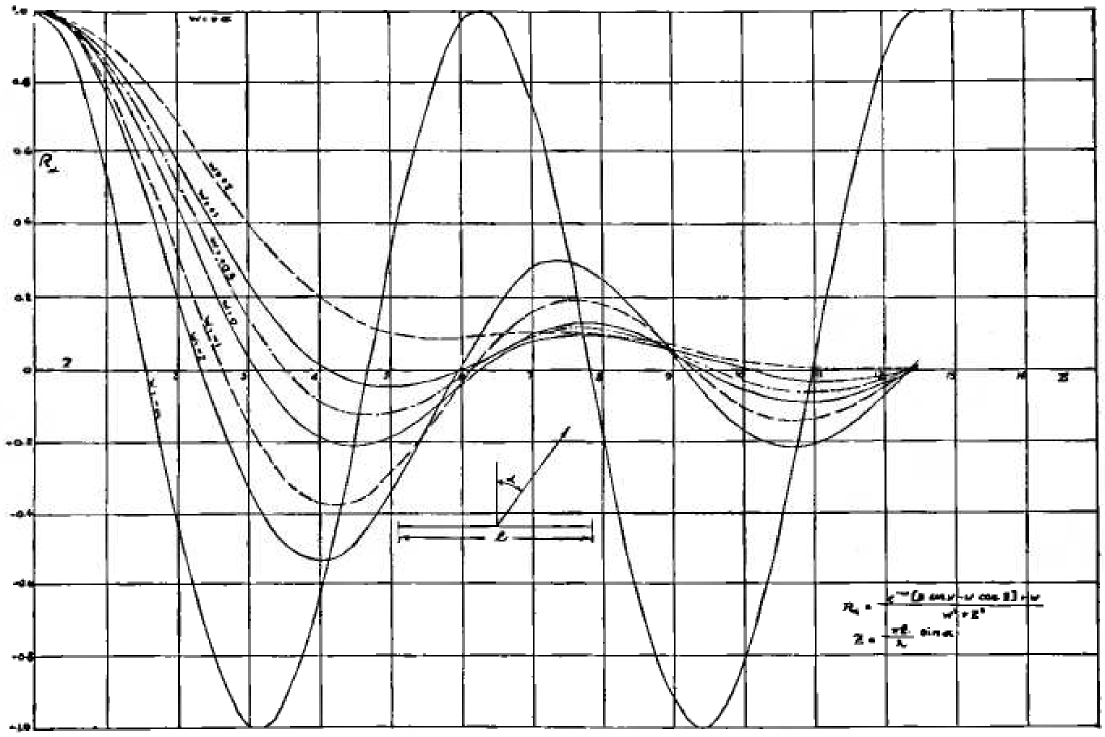

FIG. 8. *Theoretical Radiation Distribution Characteristics of Line Source with Exponential Distribution of Velocity Along Source.*

5. *Curved Line Sources*

The consideration of the radiation from a circular arc is of interest as we commonly have to deal with the directional characteristics of the waves which are constrained between flaring surfaces up to a certain distance from the source and the radiation which reaches the free air may be considered to have an arc shape as it issues from the opening. If R_α, in the plane of the arc, is determined for the case of a circular arc, a result is obtained in the form of an infinite series of Bessel's functions of ascending order. This result is not amenable to simple computation.

An approximation may be made by breaking the line up into a number of equal chords of a circular arc on each of which the intensity and phase is uniform. In this case the result takes the form:

$$R_\alpha = \frac{1}{2m+1}\left| \sum_{k=-m}^{k=+m} \cos\left[\frac{2\pi R}{\lambda}\cos(\alpha+k\theta)\right] \frac{\sin\left[\frac{\pi d}{\lambda}\sin(\alpha+k\theta)\right]}{\frac{\pi d}{\lambda}\sin(\alpha+k\theta)} \right.$$

$$\left. + i \sum_{k=-m}^{k=+m} \sin\left[\frac{2\pi R}{\lambda}\cos(\alpha+k\theta)\right] \frac{\sin\left[\frac{\pi d}{\lambda}\sin(\alpha+k\theta)\right]}{\frac{\pi d}{\lambda}\sin(\alpha+k\theta)} \right|.$$

In this equation the number of chords equals $2m+1$, the radius of the arc is R and θ is the angle subtended by any of the chords at the center of the circumscribing circle, d equals the length of one of the

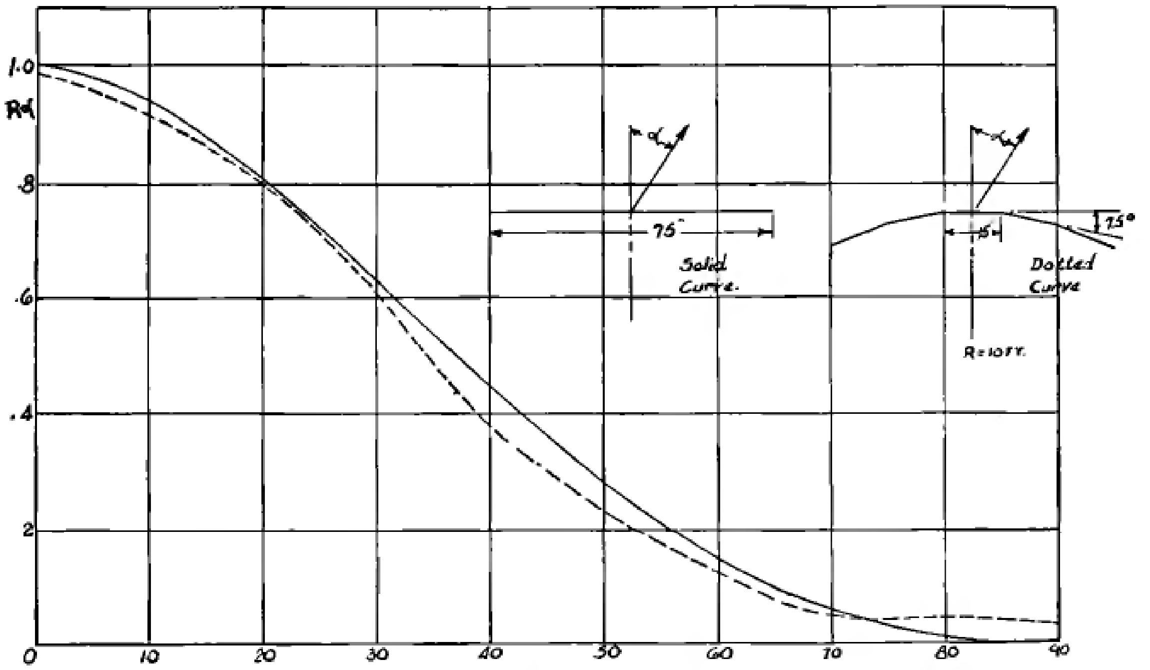

FIG. 9. *Theoretical Radiation Distribution Characteristics Combination of Line Sources (Special Case at 175 cycles).*

chords. We have plotted this result for a special case where $R=10$ ft., $d=15''$, $\theta=7\frac{1}{2}°$, at frequencies of 175 cycles, 350 cycles, 700 cycles and 1400 cycles in Figs. 9, 10, 11, 12.

For purposes of comparison we have plotted on the same sheets the characteristics which would be obtained if the chords were arranged so as to form a straight line.

These curves show that at low frequencies the characteristics are substantially the same. At higher frequencies, however, the characteristic of the circular arrangement of lines becomes broader and flatter than that of the corresponding line source, in fact, it shows a tendency to radiate uniformly throughout an angle equal to that defined by the centers of the two extreme line sources.

At low frequencies the distance that the arc is displaced from a straight line is negligible in comparison with the wave length. As the

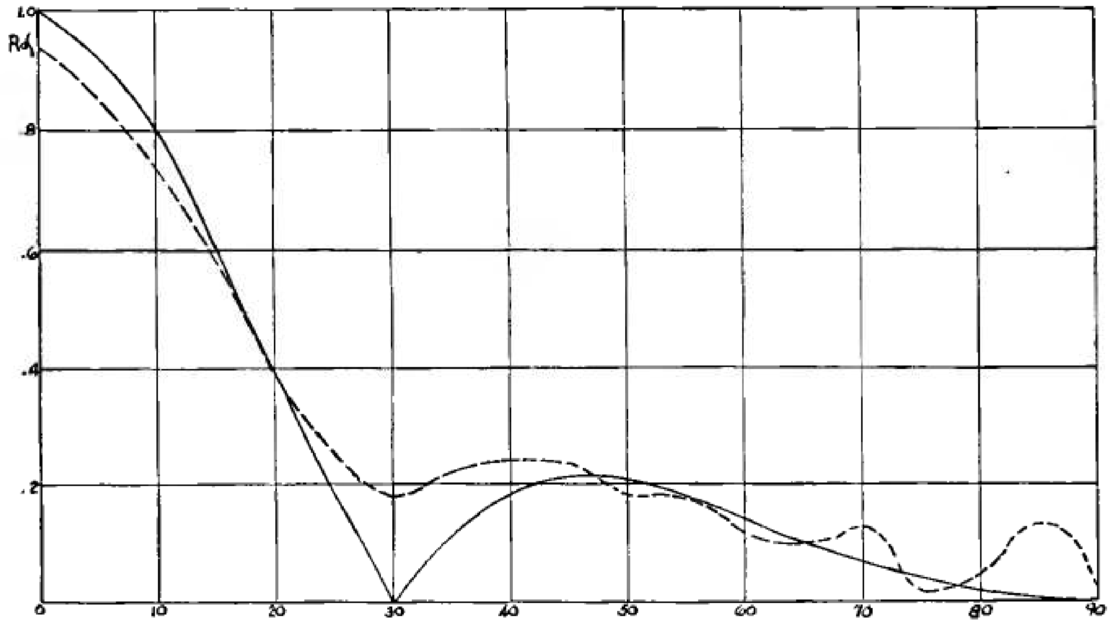

Fig. 10. *Theoretical Radiation Characteristics of Combination of Line Sources. (Special Case at 350 cycles.)*

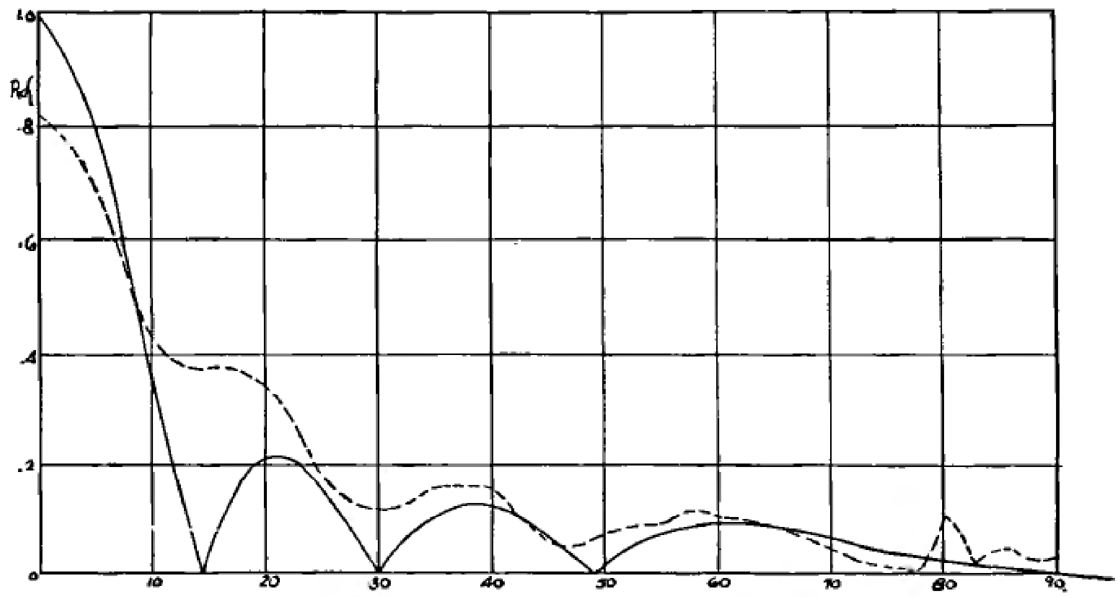

Fig. 11. *Theoretical Radiation Distribution Characteristics Combination of Line Sources (Special Case at 700 cycles).*

frequency is raised and the wave length shortens, this is no longer true. At high enough frequencies an arc should radiate almost uniformly into an angle determined by its bounding radii. By the same reasoning the series of chords will give a good approximation to the arc as long as the greatest distance from the chord to the corresponding point on the arc is small compared to the wave length of sound being radiated. In the case chosen for illustration this distance is approximately $\frac{1}{2}''$, which is small compared to the wavelength at 1400 cycles, the highest frequency for which a computation was made.

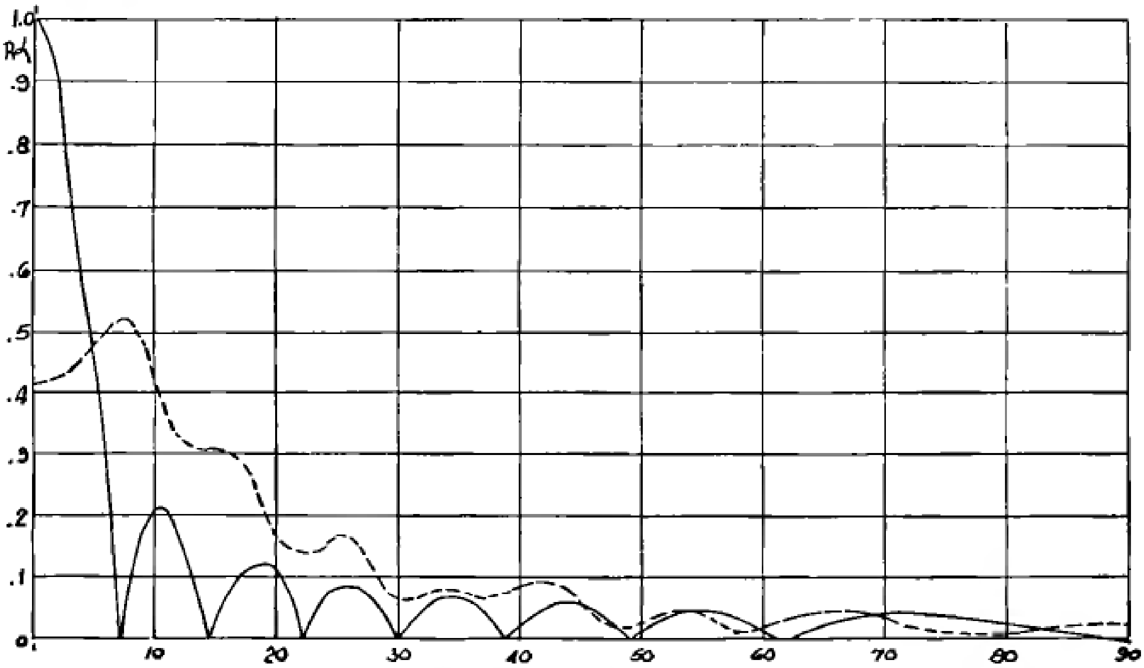

FIG. 12. *Theoretical Radiation Distribution Characteristics Combination of Line Sources (Special Case at 1400 cycles).*

6. *Plane Surface Sources*

In general, the sources which radiate sound are not lines or points, but consist of a single surface or a number of surfaces.

We shall now show how many of the results which have been obtained up to this time for points and lines may be used to determine the radiation from surfaces by a consideration of the two following general ideas.

(1) A great many surface distributions are of the form $f(v) = f_1(x) f_2(y)$ where $f(v)$ represents the distribution of intensity, and the phase distribution is represented by $F(v) = F_1(x) + F_2(y)$. In this case it may be shown that the directional characteristic is represented by:

$$R_\alpha = \frac{1}{\int f_1(x)dx \int f_2(y)dy} \left| \int f_1(x) e^{2\pi i((x\sin\alpha/\lambda)-(F_1(x)/2\pi))}dx \right.$$

$$\left. \int f_2(y) e^{2\pi i((y\sin\beta/\lambda)-(F_2(y)/2\pi))}dy \right|$$

The integration being carried out over the entire surface. (See Appendix Section C for theory). α is the angle which the line connecting the point and source make with the plane normal to the x axis, and β is the angle between the same line and the plane normal to the y axis.

It is seen that the radiation characteristic is equal to the product of

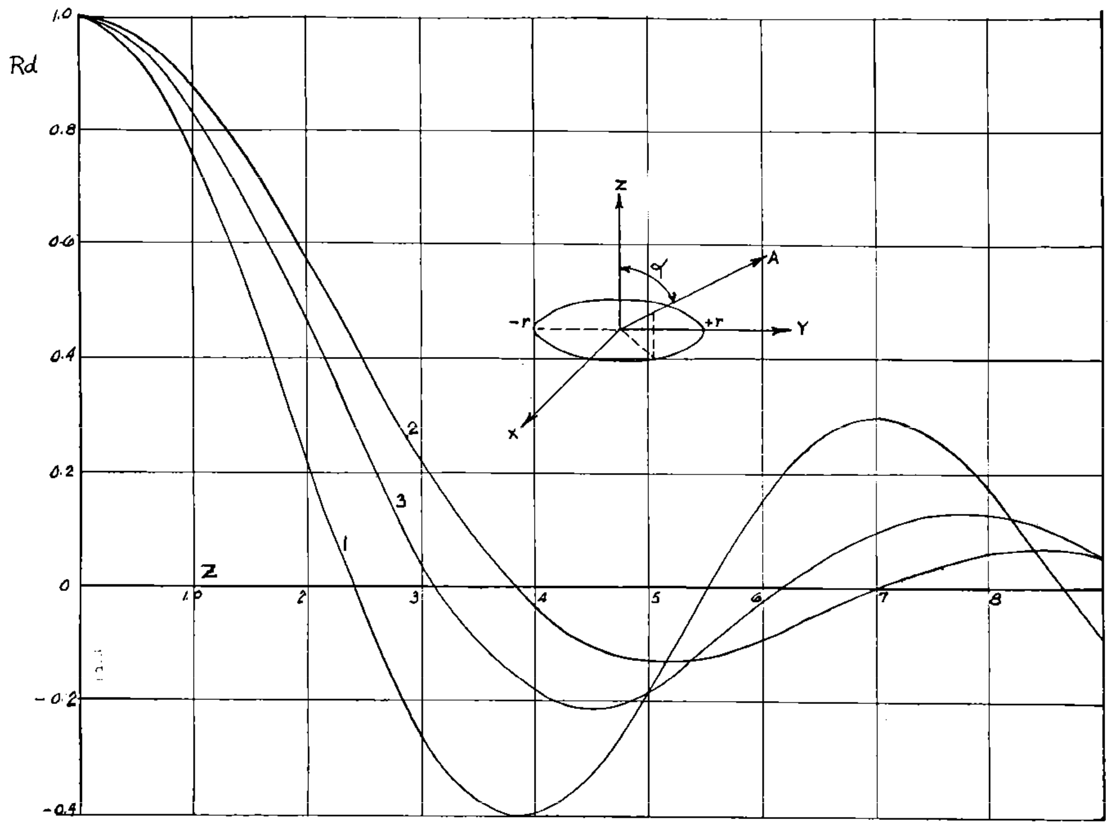

FIG. 13. *Theoretical Radiation Distribution Characteristics of*
1. *Circular Line.* $R_\alpha = J_0(Z)$
2. *Circular Surface.* $R_\alpha = 2J_1(Z)/Z$.
3. *Straight Line of Length equal to Diameter of above circle.* $R_\alpha = \dfrac{\sin Z}{Z}$ *where* $Z = \left(\dfrac{\pi d}{\lambda}\right) \sin \alpha$.

the characteristics of two mutually perpendicular line sources with intensity and phase distributions given respectively by:

$$f_1(x) \text{ and } F_1(x)$$
$$f_2(y) \text{ and } F_2(y)$$

It is interesting to compare the results obtained by this formula for a plain rectangular uniform surface distribution with that given in texts on optics for diffraction through a rectangular opening. In Drude, "Theory of Optics," page 216, an actual photograph of this diffraction phenomenom is shown. It will be noted that in the directions parallel to the sides of the opening (or source, in this case), the pattern consists of a regular series of diminishing maxima between which the intensity falls off to zero. The width of the diffraction fringes is greater in the direction in which the source is narrower. In our problem the analogues of the diffraction fringes are secondary maxima.

(2) Very often it is desirable to consider the directional characteristics of a combination of identical line or surface sources linearly arranged and equally spaced. It can be shown that the directional characteristic in this case or in the general case of a spatial distribution in which the sources are held parallel to each other, is equal to the product of the characteristic of a similar arrangement of point sources, by the individual characteristic of each separate source.

Because of its interest for the approximate calculation of the directional characteristics of cone radiators, which are in common use, the radiation from a plane circular surface source is given as developed by Stenzel[7,8]

In this case $$R_\alpha = \frac{2J_1\left(\frac{2\pi r}{\lambda}\sin\alpha\right)}{\frac{2\pi r}{\lambda}\sin\alpha}$$ Where J_1 is Bessel's function of order 1, r is the radius of the circle, and α is the angle between the line connecting the point and the source and the normal to the plane of the circle.

This function is plotted in Fig. 13, where it is compared with the characteristic of a line of equal length and a circular ring of equal diameter. The characteristic is somewhat broader than that of the uniform line source of length equal to the diameter of the circle, but has about the same form.

Expermental Results

Experimental characteristics will be shown of a number of sound radiators which approximate some of the ideal radiators which have been described.

In obtaining the experimental distribution characteristics the following procedure was followed and precautions taken:

(1) The sound source and sound measuring system, consisting of a condenser microphone with an associated amplifier were set up out of doors at a distance from all buildings. This was necessary in order that troublesome reflections might be eliminated.

(2) The condenser microphone was placed at such a distance from the sound source that the lines joining it to all points of the sound source were substantially parallel.

(3) The condenser microphone was always placed relative to the sound source and ground so that as the condenser microphone was moved around the source, the reflection effects due to the ground remained unchanged.

(4) Frequency characteristics were obtained at various points around the source. The actual radiation characteristics were determined from these curves. The maxima and minima of the radiation characteristics were obtained by interpolation on the frequency characteristics where necessary.

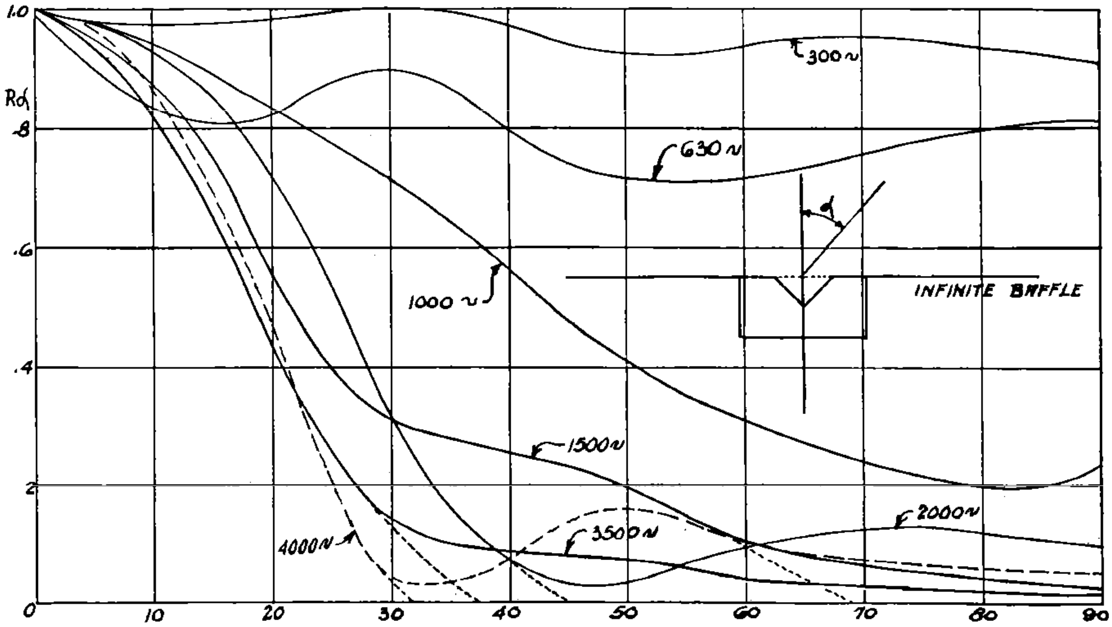

FIG. 14. *Experimental Radiation Distribution Characteristic of a 12" Cone in Infinite Baffle.*

1. *Radiation Characteristic of Single 12" Cone*

The radiation characteristic of a single 12" cone set in an infinite baffle was obtained by mounting the cone in a closed box and setting the box into the ground with the cone pointing upward so that the top of the box and the face of cone were flush with the surface of the ground.

The actual results obtained are shown in Fig. 14. Some very interesting conclusions may be drawn from a study of these curves.

Below about 700 cycles the cone behaves almost like a point source, radiating uniformly in all directions. Above this frequency, the cone radiates more or less in a beam. The sharpness of the beam increases until about 2000 cycles, above which point, the beam remains fairly uniform. This indicates that above 2000 cycles the effective size of the radiating surface becomes smaller or that the cone is no longer radiating as a plane diaphragm.

If we assume that the cone radiates in about the same way as a circular diaphragm, see Fig. 13, we can calculate in a very rough way what portion of the cone is effectively radiating at all frequencies.

From Fig. 13 for the characteristic of a circular diaphragm we see that

$$R_a = \frac{2J_1\left(\frac{2\pi r}{\lambda}\sin\alpha\right)}{\left(\frac{2\pi r}{\lambda}\sin\alpha\right)} = 0$$

for the first time when

$$\frac{2\pi r}{\lambda}\sin\alpha = 3.85$$

In Fig. (14) we have extended the experimental characteristics to meet the axis (dotted lines). We assume that the radiation actually present at greater angles is due to secondary maxima and to the fact that the cone does not exactly simulate a diaphragm.

Thus, to calculate the effective diameter of the diaphragm which can be substituted for the cone at 2000 cycles, we note that at 2000 cycles the characteristic (extended) falls off to zero for $\alpha = 45°$.

Therefore:

$$\frac{2\pi r}{\lambda}\sin\alpha = 3.85$$

$$= \frac{2\pi r}{\frac{1100}{2000}}\cos 45$$

$$= \frac{40\pi r}{\pi} = .707$$

solving for $D = 2r$, we obtain

$$D = 11.5 \text{ inches}$$

Thus, at this frequency practically the entire surface of the cone is radiating.

Proceeding in the same way, we obtain:

at 1500 cycles $D = 12''$
at 3500 " $D = 7.7''$
at 4000 " $D = 7.5''$

Thus, as the frequency is raised the effective radiating surface becomes smaller. This does not necessarily mean that only a portion of the surface is vibrating, but perhaps that piston-like motion is no longer taking place, the motion being of such a character as to approximate a piston-like motion of a smaller diameter diaphragm.

This radiator approximates the loud speakers which are ordinarily used in home radio sets with the exception of its size and the fact that

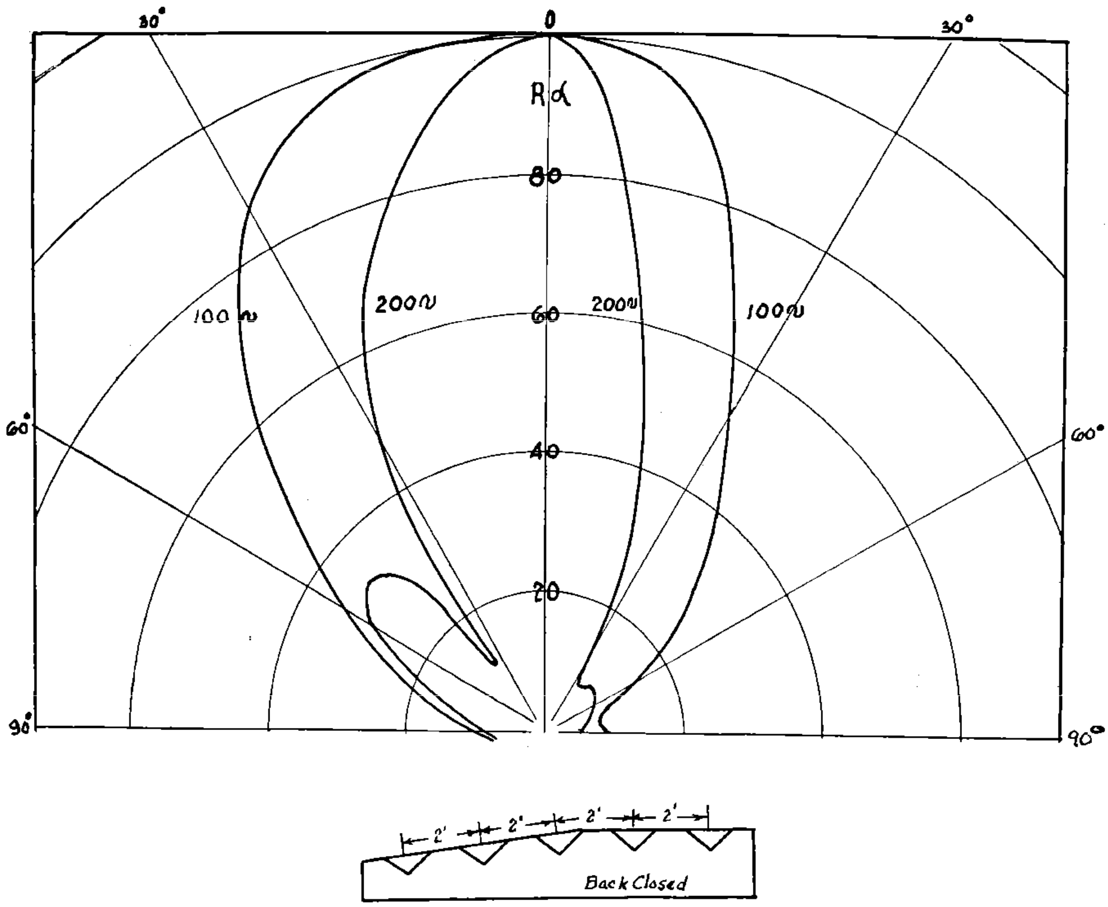

Fig. 15. *Experimental Radiation Distribution Characteristics of 5 Cone Combination in Line of Cones.*

it is placed in an infinite baffle. As is evident from the equation for radiation from a circular diaphragm the effect of reducing the size to that more normally used, will be to postpone the directional radiation to a higher frequency. The effect of placing the loud speaker in a finite baffle rather than in an infinite baffle will be to increase the directional effects somewhat at the lower frequencies, and to cause an additional number of secondary maxima which were not present in the ideal case. This will be illustrated in one of the later experimental characteristics on a slightly different form of radiator. If the desirable characteristic for a home loud speaker is that it should radiate uniformly at all frequencies, it is evident that this loud speaker is defective in radiating too

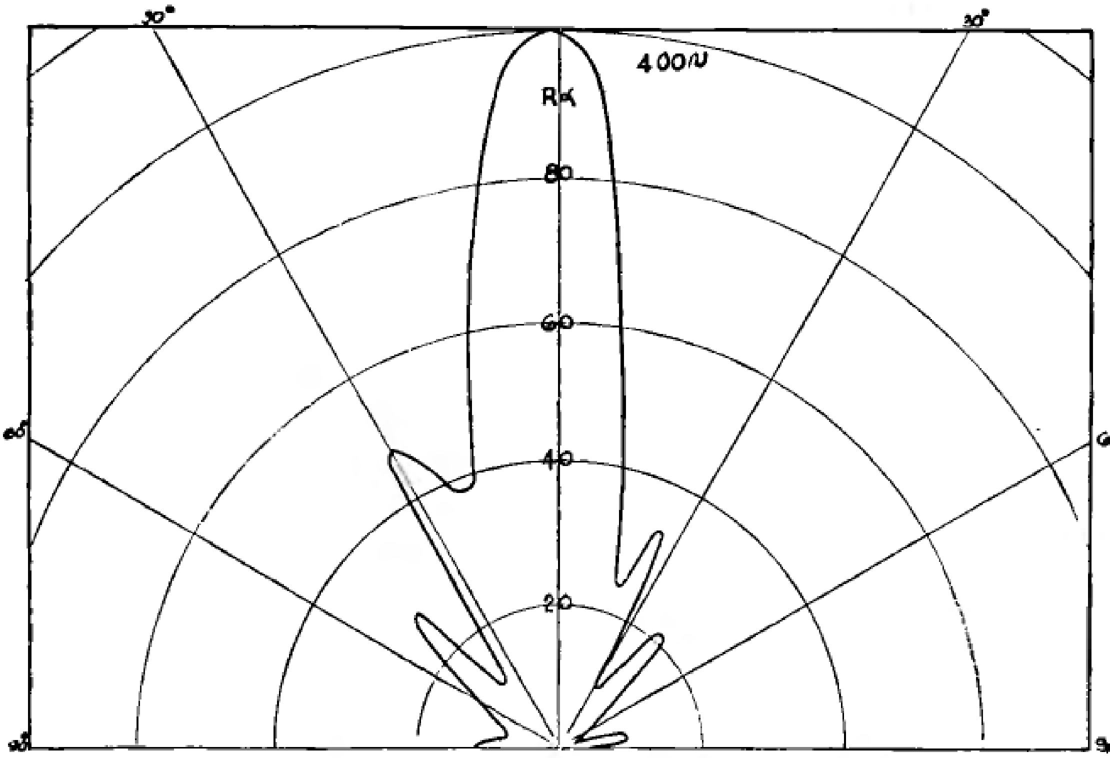

Fig. 16. *Experimental Radiation Distribution Characteristics of 5 Cone Combination in Line of Cones.*

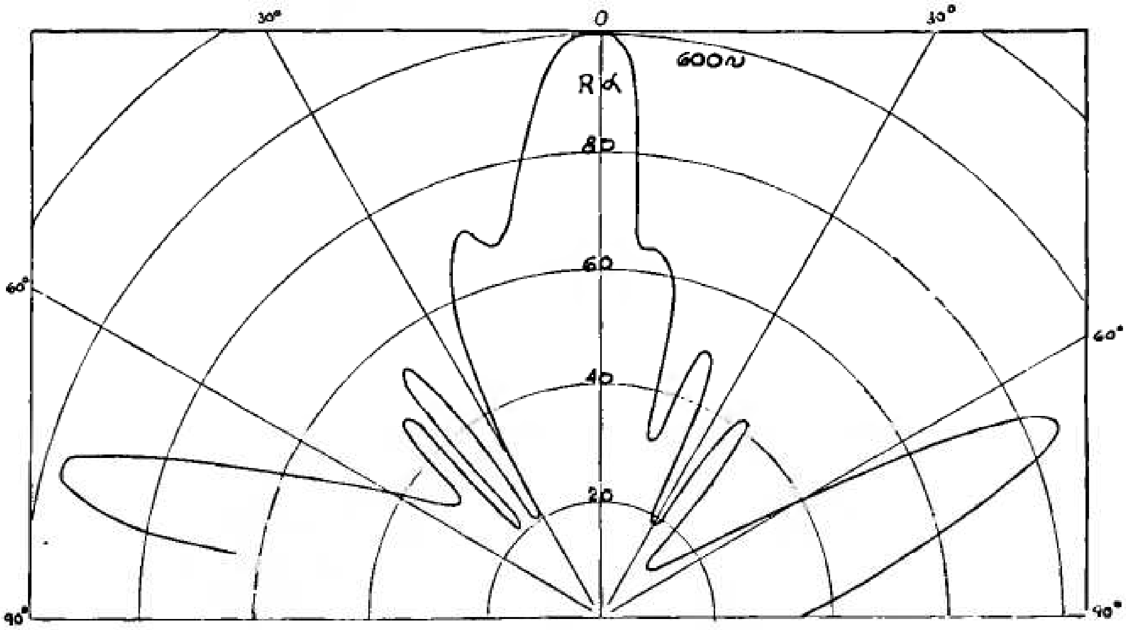

Fig. 17. *Experimental Radiation Distribution Characteristics of 5 Cone Combination in Line of Cones.*

much in the form of a beam at the higher frequencies. A method for breaking up this beam is shown in Fig 19.

A number of cones in a straight line and in contact approximate a continuous line source. As has been shown theoretically, the directional characteristic of a continuous line source contains secondary maxima of gradually decreasing intensity. On the other hand, the directional characteristic of a system of points along a line contains secondary maxima which can be as large as the primary central maximum. In order to show the extent to which a rather small separation of the cones can make the source behave like a linear arrangement of point sources,

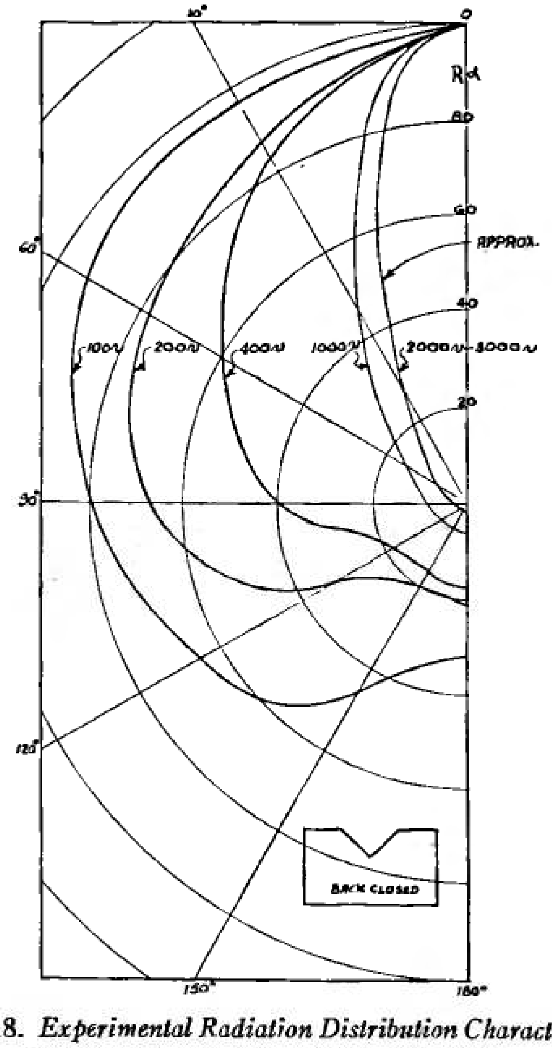

FIG. 18. *Experimental Radiation Distribution Characteristics of 5 Cone Combination Across Line of Cones.*

rather than like a continuous source, we are showing the characteristic of an experimental arrangement in which the cones are spaced apart by a distance equal to their diameters. Five cones were chosen which were arranged in a slightly unsymmetrical fashion which did not, however, affect the character of the phenomena greatly. In order to eliminate interference effects from the rear the back was closed off and partially filled with felt. The experimental characteristic in a plane passing through the line of cones is shown in Figs. 15. 16, and 17. These curves

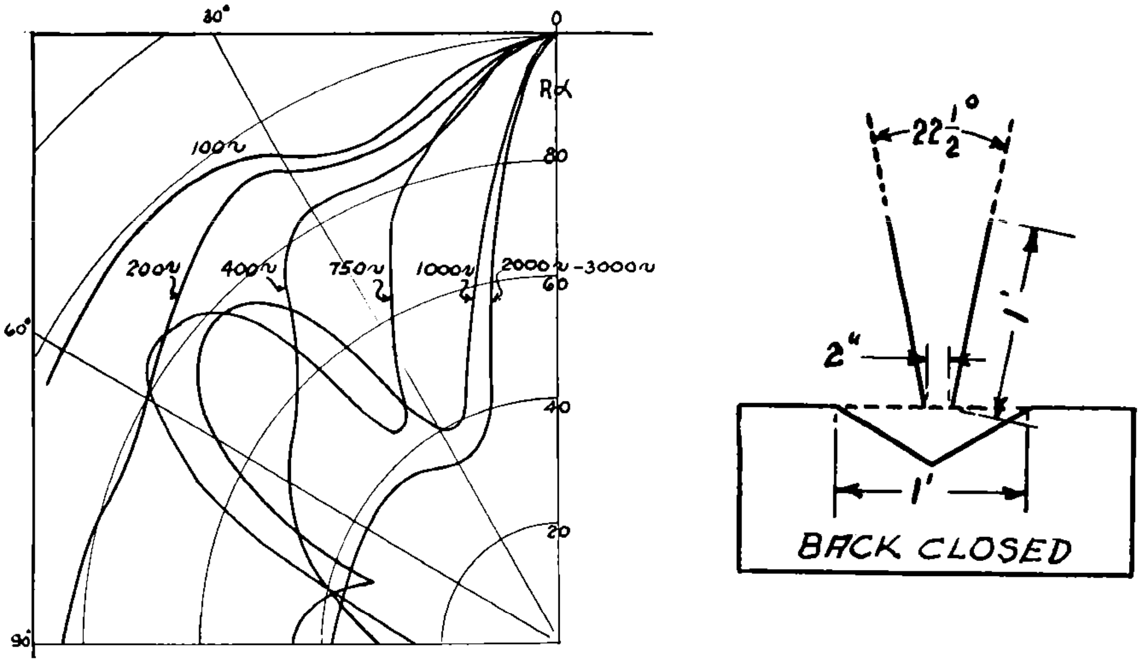

FIG. 19. *Experimental Radiation Distribution Characteristics of 5 Cone Combination Across Line of Cones. Small Reflectors Attached.*

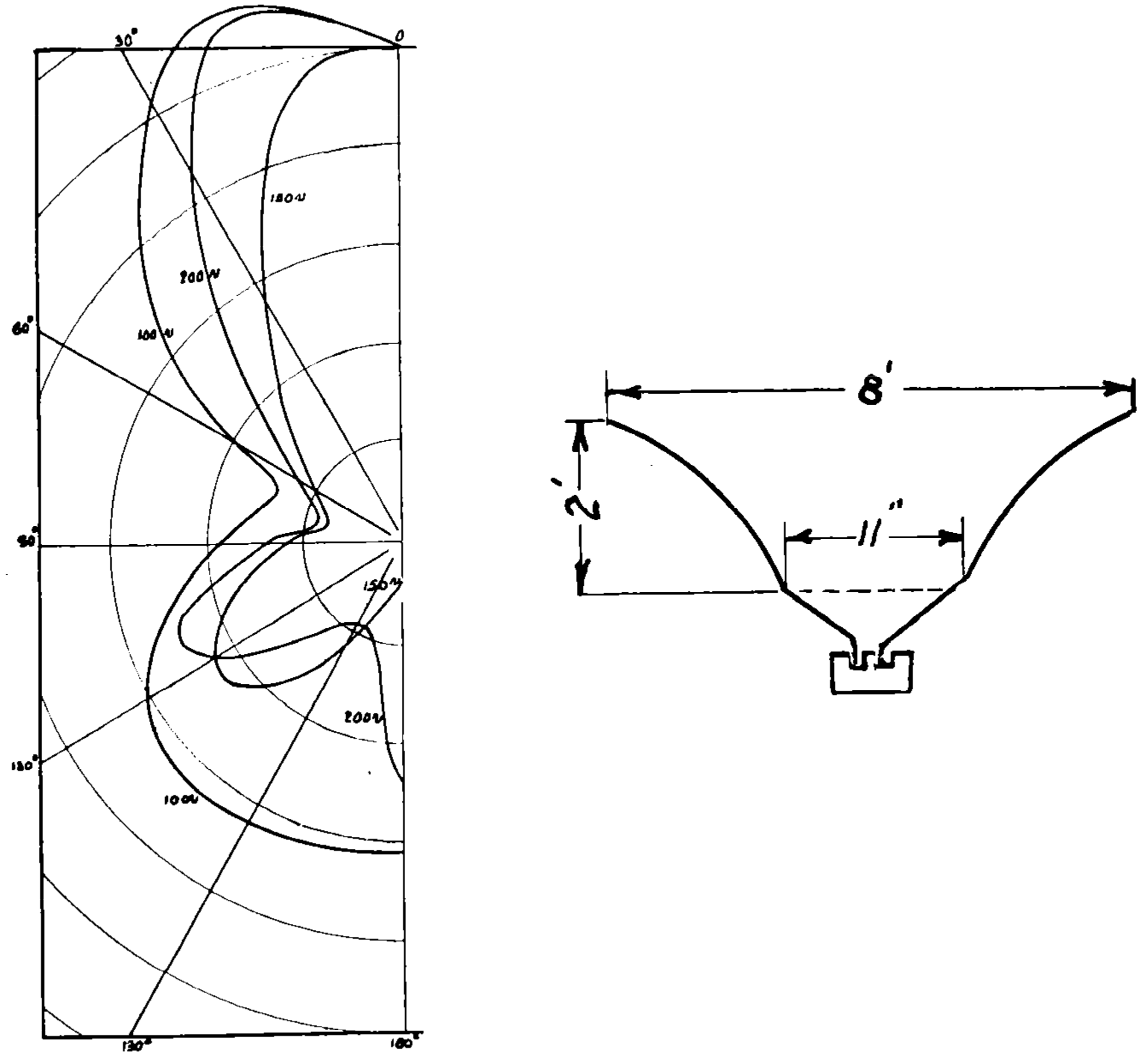

FIG. 20. *Experimental Radiation Distribution Characteristic of 6 Cone Combination Across Line of Cones. (Short Flaring Baffle.)*

may be compared with those of Figs. 1, 2, 3 and 4 for the characteristics of a linear arrangement of point sources whose distance apart is equal to the distance between the centers of the cones.

The characteristic in a plane perpendicular to the line of cones for the same arrangement is shown in Fig. 18. The fact that these do not agree exactly with the characteristics previously described for a single

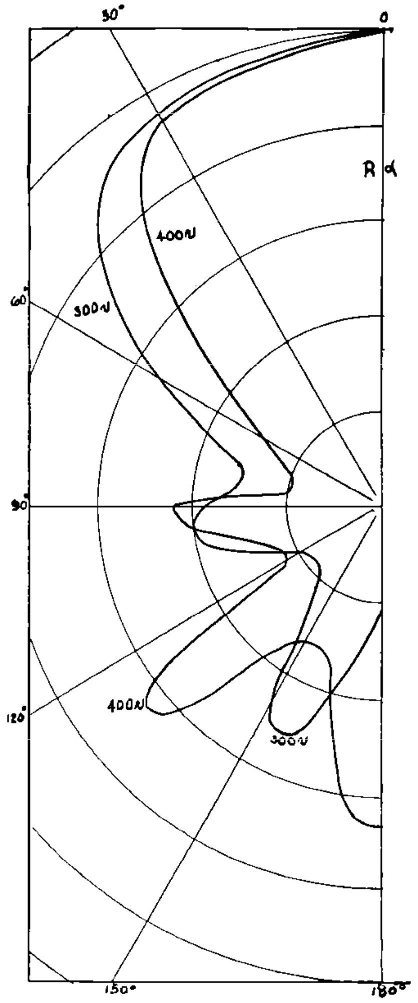

Fig. 21. *Experimetal Radiation Distribution Characteristics of 6 Cone Combination Across Line of Cones. (Short Flaring Baffle.)*

cone is explainable on the basis of (1) the characteristic of the radiation from a line of cones across the line is theoretically not the same as that of a single cone (2). The radiation from the backs of the cones was not completely shut-off. This would permit some radiation to escape from the rear, the effect of the rear radiation being to sharpen the characteristic at low frequencies, as will be shown later for an arrangement in which the rear is completely open.

At frequencies above 1,000 cycles, the beam becomes too sharp. Fig.

19 shows the effect of placing small reflectors in front of the cones. In the curves shown, the angle between these deflectors is somewhat too great, causing the radiation incident upon the deflectors to be deflected through too sharp an angle

The effect of allowing the rear radiation to escape is shown in the next series of curves. The directional characteristic of an arrangement,

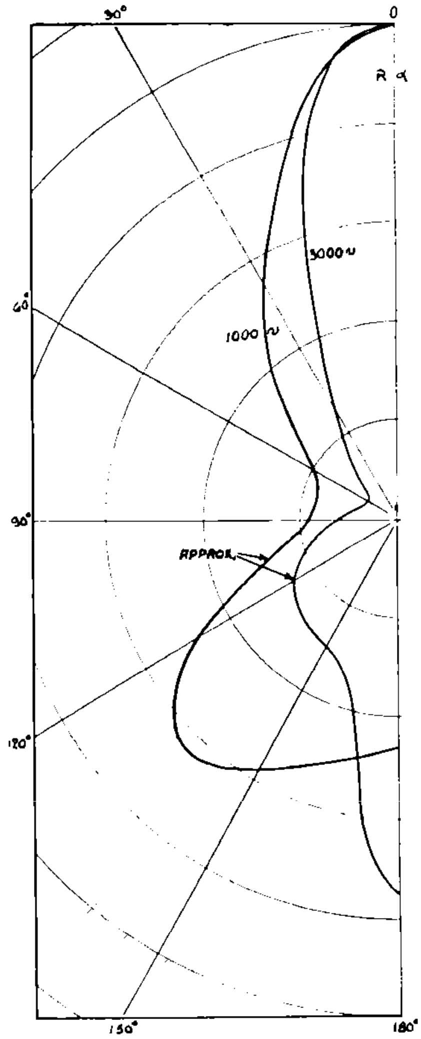

FIG. 22. *Radiation Distribution Characteristics of 6 Cone Combination Across Line of Cones. (Short Flaring Baffle.)*

consisting of six 8 inch cones in contact with a short flaring baffle on the front side, is shown in Figs. 20, 21 and 22. These curves show very well the effect of allowing sound radiation to escape from the rear of the combination. It will be noted that there is always a minimum in the neighborhood of 70° to 80°. This minimum is due to the interference of the out of phase sound waves, from the front and rear. If the combination had been symmetrical, this minmium would have occurred at 90.° Due to the shape of the baffle, however, the path length from the front to the 90° point is somewhat longer that that from the rear, so that the region of destructive interference is shifted forward slightly so as to make the minimum occur at the position shown. The somewhat higher intensity in front and the number of peaks shown in the directional characteristics to the rear, which do not similarly show up to the front, is also due to this dissymmetry. Due to the loading action of the baffle to the front side and due to the interfering effect of the loud speaker field coil magnet, the radiation to the front side is greater than that to

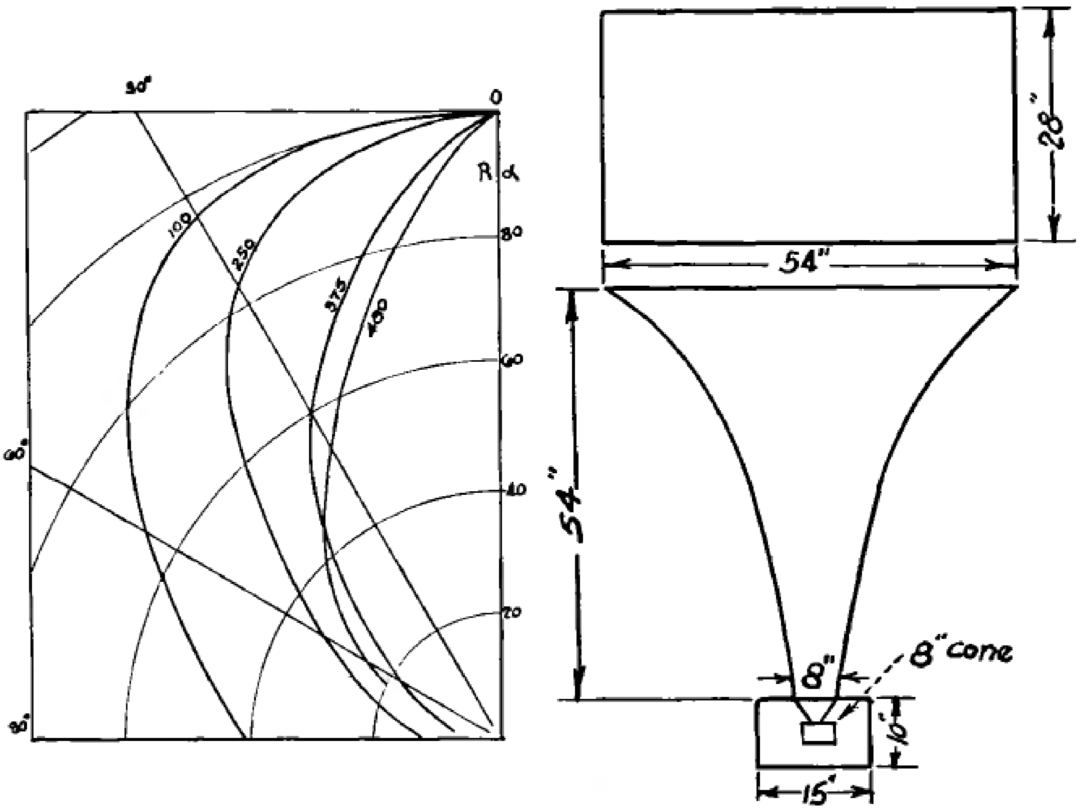

FIG. 23. *Experimental Radiation Distribution Characteristics of Directional Baffle Speaker Across Long Axis of Mouth.*

the rear. The sound from the front is attenuated by the time it reaches the rear and is almost of the same intensity as that coming from the rear of the loud speaker, thus leading to severe interference peaks. On the other hand, the sound coming from the front is little affected by the attenuated sound from the rear, which was already weaker when it started. If the rear radiation had been closed off, the directional characteristics would not have been as sharp as that shown on the front side. On the other hand, the radiation to the rear would have been much smaller. Whether this type of radiation characteristic is desirable or not, will depend largely on the place where the loud speaker is to be used. If it is to be used where good radiation to the front is required with small radiation to the sides, and where a certain amount of radi-

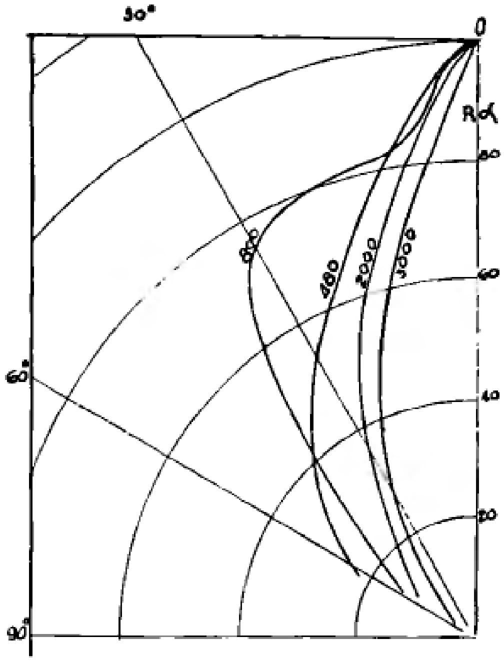

FIG. 24. *Experimental Radiation Distribution Characteristics of Directional Baffle Speaker Across Long Axis of Mouth.*

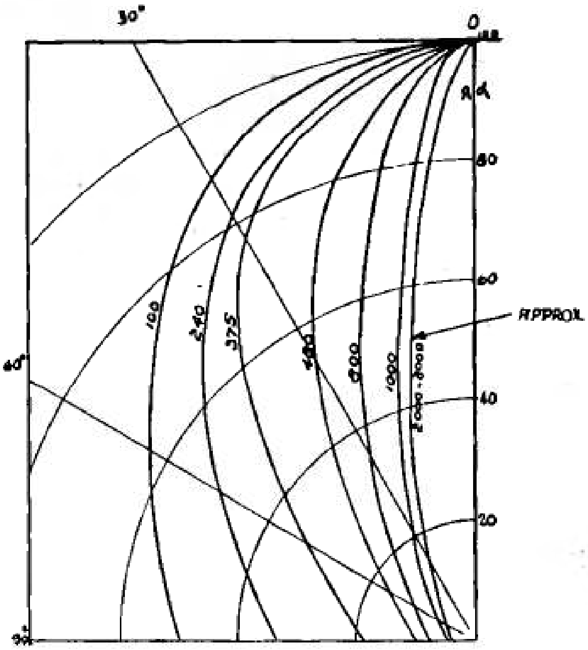

FIG. 25. *Experimental Radiation Distribution Characteristics of Directional Baffle Speaker Across Short Axis of Mouth.*

ation to the rear may be neglected, the characteristic shown is quite satisfactory.

The increased directional response which can be obtained by making a radiating surface effectively larger is shown in Figs. 23, 24, 25 for an

8″ cone with a directional baffle. A comparison of these characteristics with those for the larger 12″ cone, shows to what extent the directional effects have been increased. The characteristics across the short axis are roughly those which would be obtained from a line radiator of length somewhat shorter than the width of the mouth. The characteristic across the long flaring axis is very interesting in that between 400 and 1,000 cycles, the characteristic is quite uniform. This is probably due to the fact that at frequencies below 1,000 cycles the emerging sound is in the form of a cylindrical wave, rather than that of a plane wave. The characteristic of a source of this type is approximately the same as that due to an arrangement of line sources on a circular arc (Fig. 9, 10, 11, 12). As found above, this type of source has a fairly uniform frequency characteristic at all but very low frequencies. Measurements of sound pressure made across the mouth of the loud speaker show a definite falling off in intensity towards the edges. This explains the lack of secondary maxima and the fact that the effective size is less than the size of the mouth opening, as may be seen by a reference to Fig. 8, which shows the directional characteristics for a source with distribution which decreases towards the edges.

The characteristic shown in the last curve is the most satisfactory of those shown for theatre reproduction purposes, due to the fairly sharp uniform directional characteristics and the lack of marked secondary maxima and minima.

APPENDIX

Section A—Combinations of Point Sources

We can write for the pressure at a point A due to a harmonic point source of sound k

$$P_k = \frac{B_k}{r_k}\cos 2\pi\left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi}\right) \tag{1}$$

where

ν = frequency

λ = wavelength

V_k = distance between A and k

B_k is proportional to intensity of radiation of point source k

ϕ_k is the phase angle between the motion at k and a standard of phase.

The resultant pressure for n point sources vibrating at the same frequency and arbitrarily positioned in space is given by:

$$P = \sum_{k=1}^{k=n} P_k = \sum_{k=1}^{k=n} \frac{B_k}{r_k}\cos 2\pi\left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi}\right). \tag{2}$$

P_1 can be represented by the projection of a rotating vector, rotating with angular velocity $2\pi\nu$ upon a suitably chosen axis (see Fig. 26) The effective radiation from another point (2) at point A is represented by another vector (2) of the same period. The phase of vector (2) may differ from that of (1) due to the fact that the distance of source (1) from A is different from that of source (2) and also due to the fact that there may be a difference in phase between the sources. There will be a number of vectors n, corresponding to the number of sources.

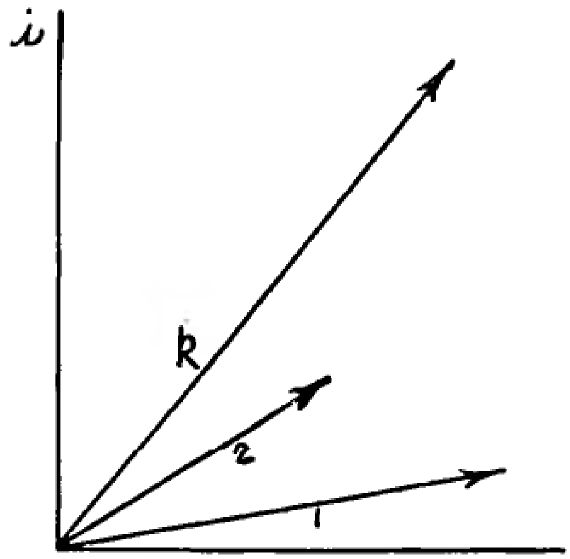

FIG. 26. *Sound Pressure Due to a System of Point Sources.*

In order to add the vectors analytically it is simplest to express each one as the sum of its projections on two mutually perpendicular axes.

$$\overline{P}_k = \frac{B_k}{r_k}\cos 2\pi\left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi}\right) + i\frac{B_k}{r_k}\sin 2\pi\left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi}\right) \tag{3}$$

$$= \frac{B_k}{r_k}e^{2\pi i(\nu t-(r_k/\lambda)-(\phi_k/2\pi))} \tag{4}$$

where $i = \sqrt{-1}$

$$= \frac{B_k}{r_k}e^{2\pi i\nu t}e^{-2\pi i((r_k/\lambda)+(\phi_k/2\pi))} \tag{5}$$

and

$$\overline{P} = \sum_{k=1}^{k=n}\overline{P}_k \tag{6}$$

$$= \sum_{k=1}^{k=n}\frac{B_k}{r_k}e^{2\pi i(\nu t-(r_k/\lambda)-(\phi_k/2\pi))}. \tag{7}$$

We shall limit our interest to the directional characteristics of radiators where the difference in distance from different parts of the radiator to the listener is small compared to the distance from radiator to listener. The expression above can then be put into the form

$$P = \frac{1}{r} \sum B_k e^{2\pi i(\nu t - (r_k/\lambda) - (\phi_k/2\pi))} \tag{7a}$$

where r_k is the distance from any one of the point sources to the listener. The absolute value of this vector which is our main interest is then given by:

$$|P| = \frac{1}{r} \left| \sum_{k=1}^{k=n} B_k e^{2\pi i(\nu t - (r_k/\lambda) - (\phi_k/2\pi))} \right| \tag{8}$$

$$= \frac{1}{r} \sqrt{\left(\sum_{k=1}^{k=n} B_k \cos 2\pi \left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi} \right) \right)^2 + \left(\sum_{k=1}^{k=n} B_k \sin 2\pi \left(\nu t - \frac{r_k}{\lambda} - \frac{\phi_k}{2\pi} \right) \right)^2} \tag{9}$$

It is evident that the maximum possible intensity at any point in space will be produced if the vectors (see Fig. 26) representing the radiation from the sources are all in phase. When this is so the absolute value of P becomes

$$|P| = \frac{1}{r} \sum_{k=1}^{k=n} B_k. \tag{10}$$

In determining the directional characteristics it is most convenient to compare the intensity at any point with the maximum possible intensity. (This maximum possible intensity does not necessarily exist at some point in space). We shall, therefore, in the following problems compute:

$$R_\alpha = \frac{1}{\frac{1}{r} \sum_{k=1}^{k=n} B_k} |P| = \frac{1}{\sum_{k=1}^{k=n} B_k} \left| \sum_{k=1}^{k=n} B_k e^{2\pi i(\nu t - (r_k/\lambda) - (\phi_k/2\pi))} \right| \tag{11}$$

The above theory will now be applied to a number of special cases.

Case (a) *Combination of N Equally Spaced Linearly Arranged Point Sources with Equal Intensity, and Moving in Phase.*

In this case let

$$\begin{aligned} B_k &= B \\ r_k &= r + (k-1)d \sin \alpha \\ \phi_k &= 0 \end{aligned} \tag{12}$$

choosing the phase angle of any source as zero phase.

d is the distance between any two successive points, α is the angle between the normal to the line of sources and line joining A with any source (see Fig. 27).

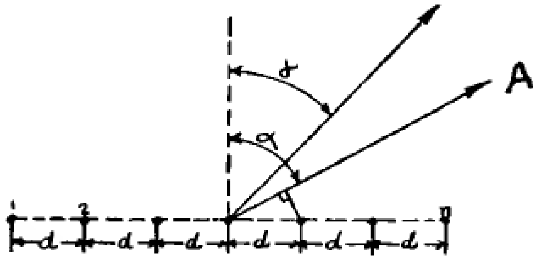

FIG. 27. *A number of equally spaced, linearly arranged point sources.*

Then

$$R_\alpha = \frac{1}{nB}\left| \sum_{k=1}^{k=n} Be^{2\pi i(\nu t-(r/\lambda))}\, e^{-2\pi i((k-1)d\sin\alpha)/\lambda}\right| \quad (13)$$

$$= \frac{1}{n}\left| \sum_{k=1}^{k=n} e^{2\pi i(\nu t-(r/\lambda))}e^{-2\pi i((k-1)d\sin\alpha)/\lambda}\right|. \quad (14)$$

Each of the terms under the Σ may be looked upon as a unit vector whose relative phase is determined by $e^{-2\pi i((k-1)d\sin\alpha)/\lambda}$ and which rotates with an angular velocity. $2\pi\nu$

Therefore, the absolute value of R_α will be given by:

$$R_\alpha = \frac{1}{n}\left| \sum_{k=1}^{k=n} e^{-2\pi i((k-1)d\sin\alpha)/\lambda}\right| \quad (15)$$

or

$$R_\alpha = \frac{1}{n}\left| \sum_{k=1}^{k=n}\left[\cos\left(2\pi\frac{(k-1)d\sin\alpha}{\lambda}\right) - i\sin\left(2\pi\frac{(k-1)d\sin\alpha}{\lambda}\right)\right]\right| \quad (16)$$

This may be represented vectorially as $1/n$ times the sum of n unit vectors inclined to each other by an angle $2\pi\,[(k-1)\,d\sin\alpha]/\lambda$

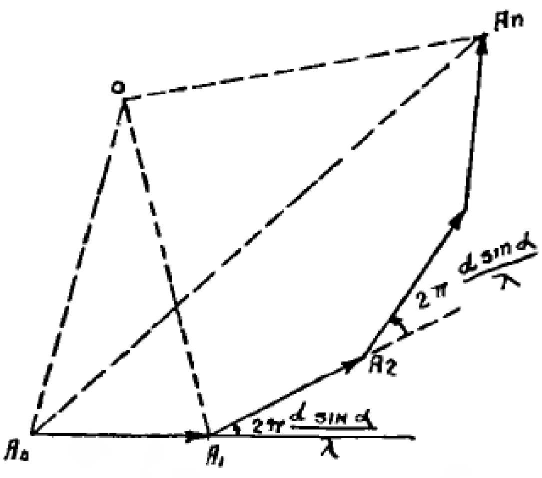

FIG. 28. *Sound pressure due to a number of equally spaced, linearly arranged point sources.*

These vectors are represented in Fig. 28 by A_0, A_1, A_1A_2,. . . $A_{n-1}A_n$ for values of k from 1 to n.

Let $OA_0 = OA_1 = OA_2 = \ldots = OA_n = P$ be the radius of the circumscribed circle.

Then

$$(A_0A_n)^2 = 2P^2 - 2P^2 \cos\left(2\pi n\frac{d \sin \alpha}{\lambda}\right) \tag{17}$$

$$= 2P^2\left[1 - \cos 2\pi\left(\frac{nd \sin \alpha}{\lambda}\right)\right] \tag{18}$$

Now

$$(A_kA_{k+1})^2 = 1 = 2P^2 + 2P^2 \cos\left(2\pi\frac{d \sin \alpha}{\lambda}\right) \tag{19}$$

$$2P^2 = \frac{1}{1 - \cos\dfrac{2\pi d \sin \alpha}{\lambda}}$$

$$(A_0A_n)^2 = \frac{1 - \cos 2\pi\dfrac{nd \sin \alpha}{\lambda}}{1 - \cos 2\pi\dfrac{d \sin \alpha}{\lambda}} \tag{20}$$

$$A_0A_n = \sqrt{\frac{1 - \cos 2\pi\dfrac{nd \sin \alpha}{\lambda}}{1 - \cos 2\pi\dfrac{d \sin \alpha}{\lambda}}} = \frac{\sin\dfrac{n\pi d \sin \alpha}{\lambda}}{\sin\dfrac{\pi d \sin \alpha}{\lambda}} \tag{21}$$

$$R_\alpha = \frac{\sin\left(\dfrac{n\pi d \sin \alpha}{\lambda}\right)}{n \sin\left(\dfrac{\pi d \sin \alpha}{\lambda}\right)} = \frac{\sin nZ}{n \sin Z} \tag{22}$$

$$Z = \frac{\pi d \sin \alpha}{\lambda} \tag{23}$$

Case (b) *Combination of N Equally Spaced Linearly Arranged Sources, Intensities Equal, Uniform Phase Shift Along Array.*

For this case

$$\begin{aligned} B_k &= B \\ r_k &= r + (k - 1)d \sin \alpha \\ \phi_k &= (k - 1)\phi \end{aligned} \tag{25}$$

where the phase of the right hand source is taken as zero phase and ϕ is the phase shift between each source.

In this case

$$R_\alpha = \frac{1}{n}\left| \sum_{k=1}^{k=n}\left[\cos 2\pi\left(\frac{(k-1)d\sin\alpha}{\lambda} + \frac{(k-1)\phi}{2\pi}\right) - i\sin 2\pi\left(\frac{(k-1)d\sin\alpha}{\lambda} + \frac{(k-1)\phi}{2\pi}\right)\right]\right| \tag{26}$$

and by analogy with the result obtained in case (a) we have

$$R_\alpha = \frac{\sin\left[n\pi\left(\frac{d\sin\alpha}{\lambda} + \frac{\phi}{2\pi}\right)\right]}{n\sin\left[\pi\left(\frac{d\sin\alpha}{\lambda} + \frac{\phi}{2\pi}\right)\right]} \tag{27}$$

$$= \frac{\sin\left[n\left(Z + \frac{\phi}{2}\right)\right]}{n\sin\left(Z + \frac{\phi}{2}\right)} = \frac{\sin nZ'}{n\sin Z'} \tag{28}$$

where:

$$Z = \frac{\pi d}{\lambda}\sin\alpha \tag{29}$$

and

$$Z' = Z + \frac{\phi}{2}.$$

In the case in which the sources are all in phase, a maximum always occurs in the direction perpendicular to the line of sources, since

for

$$\alpha = 0^\circ$$

$$\sin\alpha = 0$$

and

$$\frac{\sin\left(\frac{n\pi d}{\lambda}\sin 0^\circ\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin 0^\circ\right)} = 1 \tag{30}$$

In the case of a phase shift, a maximum will appear when

$$Z' = 0$$

That is, when

$$\frac{\pi d}{\lambda}\sin\alpha + \frac{\phi}{2} = 0$$

or

$$\alpha = \sin^{-1}\left(-\frac{\phi\lambda}{2\pi d}\right). \tag{31}$$

If γ denotes the angle of maximum radiation

$$R_\alpha = \frac{\sin\left[n\pi\frac{d}{\lambda}(\sin\alpha - \sin\gamma)\right]}{n\sin\left[\pi\frac{d}{\lambda}(\sin\alpha - \sin\gamma)\right]} \tag{32}$$

For polar diagrams of radiation characteristics of 2 and 16 linearly arranged point sources plotted for various values of b and ϕ, see reference (1).

Section B—Line Sources—Straight and Curved and Combinations Thereof

If the source is continuous, the summation in the case of a combination of discrete sources is replaced by an integration process. The integration is carried out with respect to an element of length or area depending upon whether the source is linear or a surface.

The expression representing the radiation characteristic from a line source obtained directly as a limiting case from equation 11 is:

$$R_\alpha = \frac{1}{\int f(v)dl}\left|\int f(v)e^{2\pi i(\nu t-(r_v/\lambda)-(F(v)/2\pi))}dl\right| \tag{34}$$

Here again it is useful to express the intensity at any point in terms of the maximum possible intensity.

$f(v)$ is a function representing the distribution of intensity along the line source.

$F(v)$ is a function representing the distribution of phase along the line source.

Case (a) *Straight Line Source Uniform Intensity—All Points in Phase*

For this case

$$f(v) = B$$

$$F(v) = 0$$

$$r_v = r - x\sin\alpha \qquad \text{(See Fig. 29)}$$

The value of R_α is given by:

$$R_\alpha = \frac{1}{Bl}\left| \int_{-l/2}^{+l/2} Be^{2\pi i(\nu t-(r/\lambda))}e^{-2\pi i(x \sin\alpha/\lambda)}dx \right| \tag{36}$$

The absolute value of the term on the right is given by:

$$R_\alpha = \frac{1}{l}\left| \int_{-l/2}^{+l/2} e^{2\pi i x \sin\alpha/\lambda}dx \right| \tag{37}$$

$$= \frac{\sin\left(\frac{\pi l}{\lambda}\sin\alpha\right)}{\frac{\pi l}{\lambda}\sin\alpha} = \frac{\sin Z}{Z}. \tag{38}$$

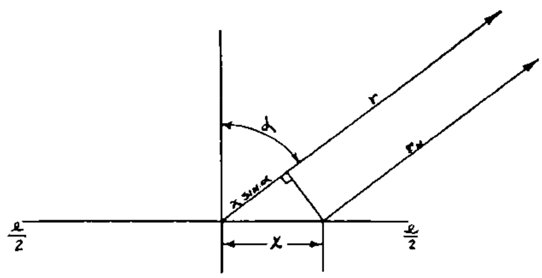

FIG. 29. *Straight line source.*

This result could have been obtained from that of a linear arrangement of point sources. For that case we obtained (see eq. 22)

$$R_\alpha = \frac{\sin\left(\frac{n\pi d}{\lambda}\sin\alpha\right)}{n\sin\left(\frac{\pi d}{\lambda}\sin\alpha\right)}$$

If we permit n to approach infinity and d to approach zero in such a way that

$$nd = l$$

we approach the line source as a limiting case. Doing this in the above equation we obtain:

$$R_\alpha = \frac{\sin\left(\frac{\pi l}{\lambda}\sin\alpha\right)}{\frac{\pi l}{\lambda}\sin\alpha}$$

which agrees with eq. (38)

Case (b) *Straight Line Source*
Uniform Intensity—Progressive Phase Shift

For the corresponding case for a linear combination of point sources we obtained (See eq. 32):

$$R_\alpha = \frac{\sin\left[n\pi\dfrac{d}{\lambda}(\sin\alpha - \sin\gamma)\right]}{n\sin\left[\pi\dfrac{d}{\lambda}(\sin\alpha - \sin\gamma)\right]} \tag{39}$$

where γ represents the angle of maximum radiation.

If we permit $n\ d$ to approach l (the length of the line source) as $n \to \infty$ and $d \to 0$, we obtain:

$$R_\alpha = \frac{\sin\left[\dfrac{\pi l}{\lambda}(\sin\alpha - \sin\gamma)\right]}{\dfrac{\pi l}{\lambda}(\sin\alpha - \sin\gamma)}. \tag{39}$$

This can also be expressed as:

$$R_\alpha = \frac{\sin\left[\pi l\left(\dfrac{\sin\alpha}{\lambda} + \dfrac{\phi}{2\pi}\right)\right]}{\pi l\left(\dfrac{\sin\alpha}{\lambda} + \dfrac{\phi}{2\pi}\right)} \tag{40}$$

where ϕ is the phase shift per unit length along the source, expressed in radians.

Case (c) *Straight Line Source.*
Non-Uniform Intensity—All Points in Phase

A particular study has been made of line radiators with distributions of intensity along the source which are symmetrical about the center. One of these will now be taken up in detail.

Exponential Distribution with Maximum or Minimum at Center
Phase Uniform

Because of the particular interest of this case and in order to show how the general theory is applied to a special case we shall work thru this case in all its details.

$$\left.\begin{array}{l} \text{Let } f(v) = e^{-mx} \text{ between } x = 0 \text{ and } x = \dfrac{l}{2} \\ \text{Let } f(v) = e^{mx} \text{ between } x = 0 \text{ and } x = -\dfrac{l}{2} \\ \text{Let } F(v) = 0 \qquad\qquad r_v = \gamma - x \sin\alpha \end{array}\right\} \tag{41}$$

The sign of m determines whether the intensity at the center is a maximum or minimum and the magnitude of m determines the rate of decay or increase of intensity along the radiator as we leave the center.

Then from equations (34) and (41) we obtain:

$$\begin{aligned} R_\alpha = & \frac{1}{\int_0^{l/2} e^{-mx} dx} \left| \int_0^{l/2} e^{-mx} e^{2\pi i(\nu t - (r/\lambda))} e^{-2\pi i(r_v/\gamma)} dx \right| \\ & + \frac{1}{\int_{-l/2}^{0} e^{mx} dx} \left| \int_{-l/2}^{0} e^{mx} e^{2\pi i(\nu t - (r/\lambda))} e^{-2\pi i(r_v/\lambda)} dx \right| \end{aligned} \tag{42}$$

The absolute value of the expression on the right hand side of eq. (42) is given by:

$$\begin{aligned} R_\alpha = & \frac{1}{\int_0^{l/2} e^{-mx} dx} \left| \int_0^{l/2} e^{-mx} e^{-2\pi i(r_v/\lambda)} dx \right| \\ & + \frac{1}{\int_{-l/2}^{0} e^{mx} dx} \left| \int_{-l/2}^{0} e^{mx} e^{-2\pi i r_v/\lambda} dx \right| \end{aligned} \tag{43}$$

Now:

$$\int_0^{l/2} e^{-mx} dx = \frac{1}{m}(1 - e^{-lm/2}) \tag{44}$$

$$\int_{-l/2}^{0} e^{mx} dx = \frac{1}{m}(1 - e^{-lm/2}) \tag{45}$$

Therefore:

$$\begin{aligned} R_\alpha = & \frac{m}{(1 - e^{-lm/2})} \left| \left\{ \int_0^{l/2} e^{-mx} \left[\cos\left(\frac{2\pi}{\lambda} x \sin\alpha\right) - i \sin\left(\frac{2\pi}{\lambda} x \sin\alpha\right) \right] dx \right\} \right| \\ & + \frac{m}{1 - e^{-lm/2}} \left| \left\{ \int_{-l/2}^{0} e^{mx} \left[\cos\left(\frac{2\pi}{\lambda} x \sin\alpha\right) - i \sin\left(\frac{2\pi}{\lambda} x \sin\alpha\right) \right] dx \right\} \right| \end{aligned} \tag{46}$$

Now:

$$\left.\begin{aligned}\int e^{qx}\cos px\,dx &= \frac{e^{qx}(q\cos px + p\sin px)}{q^2+p^2}\\ \int e^{qx}\sin px\,dx &= \frac{e^{qx}(q\sin px - p\cos px)}{q^2+p^2}\end{aligned}\right\}. \tag{47}$$

Making use of eq. (47) and substituting

$$\omega = \frac{lm}{2} \quad \text{and} \quad Z = \frac{\pi l}{\lambda}\sin\alpha$$

Equation (46) reduces to

$$R_\alpha = \frac{\omega}{1-e^{-\omega}}\left[\frac{e^{-\omega}(Z\sin Z - \omega\cos Z) + \omega}{\omega^2 + Z^2}\right] \tag{48}$$

ω is a measure of the rate of change of intensity along the source.

Substitution in Equation (48) for the limiting case in which ω approaches ∞ yields

$$R_\alpha = 1.$$

This is as it should be since a source for which $\omega = \infty$ is a point source which radiates uniformly in all directions.

In the case wherein $\omega = 0$ (the case of a uniform line source) substitution in equation (48) yields:

$$R_\alpha = \frac{\sin Z}{Z}.$$

This is in agreement with equation (38).

In the case where $\omega = -\infty$ (the case of two point sources located at the end of a line), substitution in equation (48) yields:

$$R_\alpha = \cos Z.$$

This is the result obtained when we substitute $n = 2$ in equation (22).

Section C—Surface Sources

The radiation characteristic due to a surface is given by:

$$R_\alpha = \frac{1}{\int f(v)ds}\left|\int f(v)e^{2\pi i(\nu t - (r_v/\lambda) - (F(v)/2\pi))}ds\right| \tag{49}$$

The notation is the same as above. The integration is carried out over the entire surface.

Case (a) *Plane Surface Sources*

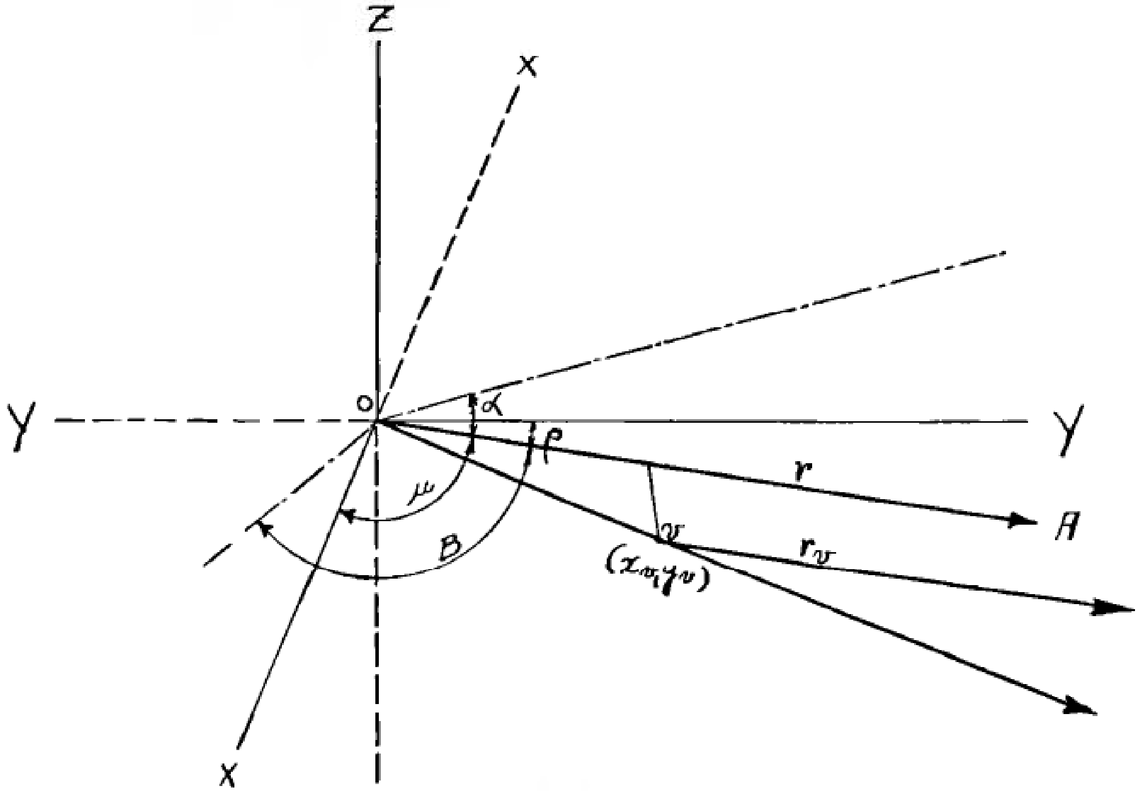

FIG. 30. *Plane surface source.*

Let the XY plane be coplanar with the plane source
Let μ = direction angle of OA with X axis
$\rho =$ " " " " " Y "
Let the coordinates of point p be given by $x_v y_v$

$$r_v = r - \overline{OV} \cos (\measuredangle AOV) \tag{50}$$

now

$$\overline{OV} = \sqrt{x^2 + y^2}$$

$$\cos (\measuredangle AOV) = \cos \mu \cos (\measuredangle VOX) + \cos \rho \cos (\measuredangle VOY) \tag{51}$$

since the cosine of the angle between any two intersecting lines in space is equal to the sum of the products of their respective direction cosines.

Now

$$\left.\begin{aligned} \cos (\measuredangle VOX) &= \frac{x}{\sqrt{x^2 + y^2}} \\ \cos (\measuredangle VOY) &= \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}\right\} \tag{52}$$

$$\overline{OP} \cos (\measuredangle AOV) = \sqrt{x^2 + y^2}\left[\frac{x}{\sqrt{x^2 + y^2}} \cos \mu + \frac{y}{\sqrt{x^2 + y^2}} \cos \rho\right] \quad (53)$$

$$= x \cos \mu + y \cos \rho$$

and

$$r_v = r - (x \cos \mu + y \cos \rho) \quad (54)$$

If α and β are the angles which $O A$ makes with the $Y Z$ and $X Z$ planes respectively

$$r_v = r - (x \sin \alpha + y \sin \beta)$$

For a plane surface source Eq. (49) becomes:

$$R_\alpha = \frac{1}{\int f(v)ds}\left|\int f(v) e^{2\pi i(\nu t-(r/\lambda)+(x \sin \alpha+y \sin \beta)/\lambda-(F(v))/2\pi)} dS\right| \quad (55)$$

The absolute value of ths expression is given by:

$$R_\alpha = \frac{1}{\int f(v)dS}\left|\int f(v) e^{2\pi i((x \sin \alpha+y \sin \beta)/\lambda-(F(v))/2\pi)} dS\right| \quad (56)$$

If $f\ (v)$ and $F\ (v)$ are of such a form that:

$$\begin{aligned} f(v) &= f_1(x) f_2(y) \\ F(v) &= F_1(x) + F_2(y) \end{aligned} \quad (57)$$

then

$$R_\alpha = \frac{1}{\int\int f_1(x) f_2(y) dy dx}$$

$$\left|\int\int f_1(x) f_2(y) e^{2\pi(ix \sin \alpha)/\lambda} e^{2\pi i(y \sin \beta)/\lambda} e^{-iF_1(x)} e^{-iF_2(y)} dy dx\right| \quad (58)$$

$$= \frac{1}{\int f_1(x)dx \int f_2(y)dy}$$

$$\left|\int f_1(x) e^{2\pi i((x \sin \alpha)/\lambda-(F_1(x))/2\pi)} dx \int f_2(y) e^{2\pi i((y \sin \beta)/\lambda-(F_2(y))/2\pi} dy\right| \quad (59)$$

This is the most general expression for the radiation characteristic of a plane surface source for which

$$\begin{aligned} f(v) &= f_1(x)f_2(y) \\ \text{and} \quad F(v) &= F_1(x) + F_2(y). \end{aligned} \tag{57}$$

Eq. (58) shows that the radiation distribution characteristic of a surface source with an intensity and phase distribution defined by eq. (57) is equal to the product of the radiation characteristics of two mutually perpendicular line sources along each of which the intensity and phase distributions are respectively given by:

$$f_1(x) \text{ and } F_1(x)$$
$$f_2(y) \text{ and } F_2(y).$$

Case (b) *Rectangular Source—Uniform Phase and Intensity*

This is really a special case of case (a) just described. From the above conditions

$$\begin{aligned} f(v) &= f(x_1 y_1) = B \\ F(v) &= F(x_1 y_1) = 0 \end{aligned} \left\{ \begin{aligned} &\text{from } x = -\frac{a}{2} \text{ to } x = +\frac{a}{2} \\ &\quad y = -\frac{b}{2} \text{ to } y = +\frac{b}{2} \end{aligned} \right\}$$

$f(v) = f(x_1, y_1)$ may be put in the form $f_1(x) f_2(y)$, where $f_1(x) = B/g$ $f_2(y) = gB$, where g is any constant, and $F(v) = F(x_1, y_1)$ into $F_1(x) + F_2(y)$ where $F_1(x) = 0$, $F_2(y) = 0$.

Using the theorem which has just been proved in eq. (59), the directional characteristics of the plane rectangular source with uniform intensity and phase should be equal to the product of the characteristics of two line sources at right angles to each other and on each of which the intensity and phase is uniform.

Making use of the result for the linear source which was developed in eq. (38)

$$R_\alpha = \frac{\sin Z}{Z} \frac{\sin w}{w}$$

where

$$Z = \frac{\pi a}{\lambda} \sin \alpha$$

$$w = \frac{\pi b}{\lambda} \sin \beta.$$

This solution is identical with that for the diffraction of light thru a rectangular aperture when the source and screen are at comparatively large distances from the aperture.