Chapter 7

HARMONIC DISTORTION

Semiconductor devices are inherently nonlinear. For example, Bipolar transistors in forward active region exhibit an exponential relationship between the collector current and the base-emitter voltage, while in saturated MOS transistors the drain current approximately depends on the square of the gate-source voltage. Therefore, circuits made up with transistors or, more generally, with real active components exhibit a certain amount of nonlinearity, and this means that the relationship between their input and the output variables is not so ideally linear as assumed in the previous chapters. Usually, active devices used for analog signal processing applications are operated in a quasi-linear region. Thus the linearity assumption is almost verified especially when signals with small amplitude are processed. However, designers are asked to evaluate the limits of the linear approximation or to characterise the effects of nonlinear distortion in circuits and systems used as linear blocks [S99][1]. To achieve these targets harmonic distortion analysis is customary employed.

Consider the open loop amplifier A_{NL} in Fig. 7.1 with its DC *nonlinear* transfer characteristic $x_o=A_{NL}(x_i)$. When nonlinearities are small, that is the transcharacteristic is characterised by gradual slope changes, the circuit is said to operate under *low-distortion conditions*[2]. This implies, in other words, that transistors do not leave the active region, and small-signal analysis can be used to produce meaningful results. Harmonic distortion in this case is usually calculated with the series expansion of the nonlinear DC

[1] Linear distortion arises in a linear amplifier which has a non constant frequency response in the frequency domain [S99].

[2] For a rigorous definition of the low-distortion condition see [OS93].

transfer characteristic. Let us assume that it is well represented by the first three power terms

$$x_o = A_{NL}(x_i) \approx a_1 x_i + a_2 x_i^2 + a_3 x_i^3 \tag{7.1}$$

Assuming now the incremental input voltage be a pure sinusoidal tone $x_i = X_i \cos(\omega t)$, the output signal becomes[3]

$$x_o = b_0 + b_1 \cos(\omega t) + b_2 \cos(2\omega t) + b_3 \cos(3\omega t) \tag{7.2}$$

where terms b_i are

$$b_0 = \frac{a_2}{2} X_i^2 \tag{7.3}$$

$$b_1 = a_1 X_i + \frac{3}{4} a_3 X_i^3 \tag{7.4}$$

$$b_2 = \frac{a_2}{2} X_i^2 \tag{7.5}$$

$$b_3 = \frac{a_3}{4} X_i^3 \tag{7.6}$$

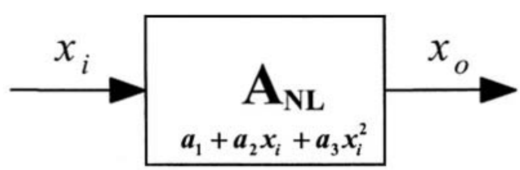

Fig. 7.1. Nonlinear open-loop amplifier.

Due to the amplifier nonlinearity, the ideal sinusoid at the input changes its shape at the output. Indeed, the output signal is a superposition of a constant term, represented by the coefficient b_0, a sinusoidal waveform with a frequency equal to that at the input multiplied by the coefficient b_1

3 Remember that $\cos^2 x = \frac{1 + \cos 2x}{2}$ and $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

(*fundamental* component), and two other sinusoidal waveforms having a frequency twice and three times greater than that of the input signal, multiplied by the coefficients b_2 and b_3, respectively (*second* and *third harmonic* components). To outline the weight of the harmonics, the harmonic distortion factors are defined as given below [S70]

$$HD_2 = \frac{|b_2|}{|b_1|} \approx \frac{1}{2}\frac{a_2}{a_1} X_i \tag{7.7a}$$

$$HD_3 = \frac{|b_3|}{|b_1|} \approx \frac{1}{4}\frac{a_3}{a_1} X_i^2 \tag{7.7b}$$

where the gain compression [MW95], which arises in term b_1 and is due to coefficient a_3, have been neglected. It is worth noting that the harmonic factors increase with the input amplitude.

In order to allow a simple comparison with the closed-loop cases that will be developed in the following paragraphs, the harmonic distortion factors can be also referred to the amplitude of the output fundamental component, $X_o \approx a_1 X_i$

$$HD_2 = \frac{1}{2}\frac{a_2}{a_1^2} X_o \tag{7.8a}$$

$$HD_3 = \frac{1}{4}\frac{a_3}{a_1^3} X_o^2 \tag{7.8b}$$

Of course, the two above equations can be used to compare the linearity performance of two different amplifiers but at the same (fundamental) output signal level.

Alternatively, we can represent the input signal by the expression $x_i = X_i e^{j\omega t}$ and the output signal, through (7.1), becomes $x_o = c_1 e^{j\omega t} + c_2 e^{j2\omega t} + c_3 e^{j3\omega t} = a_1 X_i e^{j\omega t} + a_2 X_i^2 e^{j2\omega t} + a_3 X_i^3 e^{j3\omega t}$. Thus, to obtain the same distortion factors as in (4) we have to define $HD_2 = \frac{1}{2}\frac{|c_2|}{|c_1|}$ and $HD_3 = \frac{1}{4}\frac{|c_3|}{|c_1|}$. As we will show this representation is useful to characterise nonlinear systems in the frequency domain.

7.1 HARMONIC DISTORTION AT LOW FREQUENCY

In this section we shall analyse the influence of feedback on harmonic distortion for low-frequency input signals. In other words, we consider the input signal frequency lower than the cut-off frequency of the loop gain, which can be therefore assumed constant, i.e. $T(j\omega) = T_0$.

7.1.1 Nonlinear Amplifier with Linear Feedback

The classical theory of feedback amplifiers asserts that negative feedback reduces harmonic distortion by the loop-gain [GM93], [LS94]. Let us consider the same amplifier in Fig. 7.1 characterised by the same nonlinear function given in (7.1), and feed a fraction f of the output signal back to the input, as shown in Fig. 7.2. This means to close the amplifier in loop with a linear feedback, f, and obtaining a return ratio T_o equal to fa_1. It is well known that the harmonic distortion terms given by (7.7a) and (7.7b) are reduced by the factors $(1+T_o)^2$ and $(1+T_o)^3$, respectively. Alternatively, we obtain a reduction by a factor $(1+T_o)$ on the harmonic distortion factors referred to the output signal magnitude.

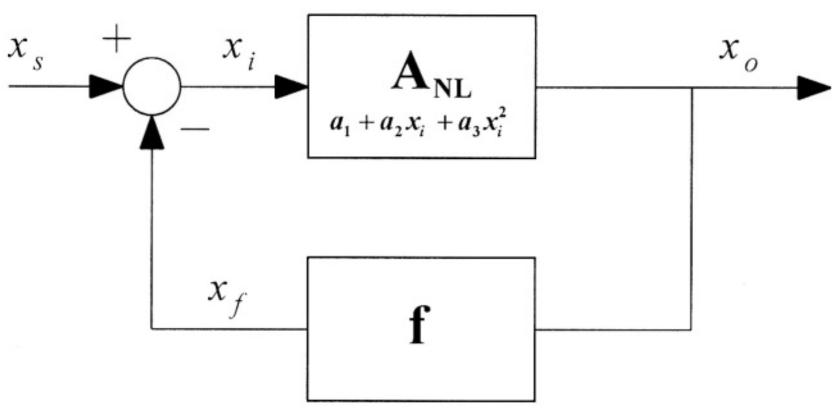

Fig. 7.2. Nonlinear amplifier with linear feedback.

Following the approach described in [PM91], a more accurate result of the harmonic distortion factors of a closed-loop amplifier can be obtained. Indeed, for the feedback amplifier in Fig. 7.2 we have

$$x_i = x_s - x_f = x_s - fx_o \tag{7.9}$$

hence relationship (7.1) can be rewritten as

$$x_o = a_1(x_s - fx_o) + a_2(x_s - fx_o)^2 + a_3(x_s - fx_o)^3 \tag{7.10}$$

The output signal can be expressed as a new power series with the source signal as independent variable

$$x_o \approx a'_1 x_i + a'_2 x_i^2 + a'_3 x_i^3 \tag{7.11}$$

where coefficients a'_1, a'_2, and a'_3 can be obtained by interpreting the power series as a Taylor's series

$$x_o = \left.\frac{dx_o}{dx_s}\right|_{0,0} x_s + \frac{1}{2}\left.\frac{d^2x_o}{dx_s^2}\right|_{0,0} x_s^2 + \frac{1}{6}\left.\frac{d^3x_o}{dx_s^3}\right|_{0,0} x_s^3 \tag{7.12}$$

Taking the derivatives of (7.10) and considering that $x_o = 0$ when $x_s = 0$, we obtain

$$a'_1 = \frac{a_1}{1+T_o} \tag{7.13}$$

$$a'_2 = \frac{a_2}{(1+T_o)^3} \tag{7.14}$$

$$a'_3 = \frac{a_3(1+fa_1) - 2a_2^2 f}{(1+fa_1)^5} \tag{7.15}$$

that through relationships (7.7) and (7.8) lead to

$$HD_{2fl} = \frac{1}{2}\frac{a_2}{a_1}\frac{1}{(1+T_o)^2}X_s = \frac{1}{2}\frac{a_2}{a_1^2}\frac{1}{1+T_o}X_o \tag{7.16}$$

$$HD_{3fl} = \frac{1}{4}\frac{a_3}{a_1}\frac{1-\dfrac{2fa_2^2}{a_3(1+T_o)}}{(1+T_o)^3}X_s^2 = \frac{1}{4}\frac{a_3}{a_1^3}\frac{1-\dfrac{2fa_2^2}{a_3(1+T_o)}}{1+T_o}X_o^2 \qquad (7.17)$$

in which subscript "*fl*" stands for *linear feedback.*

By inspection of (7.17) we see that for amplifiers where coefficient a_3 is negligible, the third harmonic is still determined by a_2. Moreover, the third harmonic distortion can be minimised if

$$\frac{f}{1+T_o} = \frac{a_3}{2a_2^2} \qquad (7.18)$$

and for $T_o >> 1$ (7.17) and (7.18) simplify to

$$HD_{3fl} = \frac{1}{4}\frac{a_3}{a_1}\frac{1-\dfrac{2a_2^2}{a_1 a_3}}{(1+T_o)^3}X_s^2 = \frac{1}{4}\frac{a_3}{a_1^3}\frac{1-\dfrac{2a_2^2}{a_1 a_3}}{1+T_o}X_o^2 \qquad (7.19a)$$

$$\frac{2a_2^2}{a_1 a_3} = 1 \qquad (7.19b)$$

7.1.2 Nonlinear Amplifier with Nonlinear Feedback

When also the feedback network is made up of active components (for instance, when MOS transistors working in triode region are employed as feedback elements instead of pure linear resistances, [IF94]), the feedback network cannot be considered ideally linear as previously done. Evaluation of the distortion of a feedback amplifier where both the amplifier and the feedback network introduce substantial nonlinearities was carried out in [PP981], and is developed in the following.

First, consider the nonlinear behaviour of the feedback path according to

$$x_f = F_{NL}(x_{out}) = f_1 x_{out} + f_2 x_{out}^2 + f_3 x_{out}^3 \qquad (7.20)$$

The input signal, x_s, can be written as

$$x_s = x_i + x_f \qquad (7.21)$$

hence, after substituting (7.1) into (7.20), and again substituting the resulting equation into (7.21), we get

$$x_s = (1 + f_1 a_1)x_i + (f_1 a_2 + f_2 a_1^2)x_i^2 + (f_1 a_3 + 2 f_2 a_1 a_2 + f_3 a_1^3)x_i^3 \qquad (7.22)$$

We can invert a nonlinear function, $x_s(x_i)$, represented by a power series up to the third-order term

$$x_s(x_i) = c_1 x_i + c_2 x_i^2 + c_3 x_i^3 \qquad (7.23)$$

into

$$x_i(x_s) = K_1 x_s + K_2 x_s^2 + K_3 x_s^3 \qquad (7.24)$$

by using the conversion formulas [KO91], [WM95] reported below

$$K_1 = \frac{1}{c_1} \qquad (7.25)$$

$$K_2 = -\frac{c_2}{c_1^3} \qquad (7.26)$$

$$K_3 = \frac{1}{c_1^3}\left[-\frac{c_3}{c_1} + 2\left(\frac{c_2}{c_1}\right)^2\right] \qquad (7.27)$$

Therefore, x_i is given by

$$x_i = \frac{1}{1 + f_1 a_1} x_s - \frac{f_1 a_2 + f_2 a_1^2}{(1 + f_1 a_1)^3} x_s^2 - \frac{1}{(1 + f_1 a_1)^3}\left[\frac{f_1 a_3 + 2 f_2 a_1 a_2 + f_3 a_1^3}{1 + f_1 a_1} - 2\left(\frac{f_1 a_2 + f_2 a_1^2}{1 + f_1 a_1}\right)^2\right] x_s^3 \qquad (7.28)$$

and combining (7.28) with (7.1), taking only the first three power terms, we get

$$x_o = a_1K_1x_s + \left(a_1K_2 + a_2K_1^2\right)x_s^2 + \left(a_1K_3 + 2a_2K_1K_2 + a_3K_1^3\right)x_s^3 \tag{7.29}$$

In conclusion, being $T_o = a_1f_1$, the second and third harmonic distortion factors are

$$HD_{2f} = \frac{1}{2}\left(\frac{K_2}{K_1} + \frac{a_2}{a_1}K_1\right)X_s = \frac{1}{2}\frac{a_2 - f_2a_1^3}{a_1(1+T_o)^2}X_s = \frac{1}{2}\frac{a_2 - f_2a_1^3}{a_1^2(1+T_o)}X_o \tag{7.30}$$

$$HD_{3f} = \frac{1}{4}\left(\frac{K_3}{K_1} + 2\frac{a_2}{a_1}K_2 + \frac{a_3}{a_1}K_1^2\right)X_s^2 =$$

$$= \frac{1}{4}\frac{a_3}{a_1}\frac{1 - f_3\frac{a_1^4}{a_3} - \frac{2f_1a_2^2 + 4f_2a_1^2a_2 - 2f_2^2a_1^5}{a_3(1+T_o)}}{(1+T_o)^3}X_s^2 = \tag{7.31}$$

$$= \frac{1}{4}\frac{a_3}{a_1^3}\frac{1 - f_3\frac{a_1^4}{a_3} - \frac{2f_1a_2^2 + 4f_2a_1^2a_2 - 2f_2^2a_1^5}{a_3(1+T_o)}}{1+T_o}X_o^2$$

A more compact form of the second and third harmonic distortion coefficients can be obtained considering that the return ratio, T_o, is usually much greater than one. Hence, after normalising the amplifier terms a_2, a_3 to the amplifier gain a_1, and the feedback terms f_2, f_3 to the feedback linear term, f_1, (by defining a_{2N}, $= a_2/a_1$, $a_{3N} = a_3/a_1$, $f_{2N} = f_2/f_1$ and $f_{3N} = f_3/f_1$) we get

$$HD_{2f} = \frac{1}{2}\left(\frac{1}{Ta_1}a_{2N} - f_{2N}\right)X_o \tag{7.32}$$

$$HD_{3f} = \frac{1}{4}\left[\frac{1}{Ta_1^2}\left(a_{3N} - 2a_{2N}^2\right) - \left(f_{3N} - 2f_{2N}^2\right) - 4\frac{1}{Ta_1}a_{2N}f_{2N}\right]X_o^2 \tag{7.33}$$

By inspection of (7.32) and (7.33), it is apparent that feedback does not reduce the nonlinearity of the feedback network. Thus, we cannot obtain an amplifier having a nonlinearity lower than that of the feedback network, and even small nonlinearity terms of the feedback networks cannot be neglected, but they must be taken into account.

In order to evaluate the different weight between the nonlinearity of the amplifier and that of the feedback network, it is useful to write the two coefficients when the amplifier is linear (i.e., with $a_2 = 0$ and $a_3 = 0$)

$$HD_{2fn} = -\frac{1}{2}\frac{f_2 a_1^2}{(1+T_o)^2}X_s = -\frac{1}{2}\frac{f_2 a_1}{(1+T_o)}X_o \tag{7.34}$$

$$HD_{3fn} = -\frac{1}{4}\frac{f_3 a_1^3 - \frac{2f_2^2 a_1^4}{(1+T_o)}}{(1+T_o)^3}X_s^2 = -\frac{1}{4}\frac{f_3 a_1 - \frac{2f_2^2 a_1^2}{(1+T_o)}}{1+T_o}X_o^2 \tag{7.35}$$

As expected, comparison of (7.34) and (7.35) with (7.16) and (7.17), which refer to the case of nonlinear amplifier with linear feedback, shows that the feedback path is more critical than the forward path. Indeed, assuming the nonlinearity for both the amplifier and the feedback network to be equal, which means $a_2 = f_2$ and $a_3 = f_3$, for the same output magnitude, relationship (7.16) is lower than (7.34) by a factor a_1^2, and relationship (7.17) is lower than (7.35) by about a_1^4. Moreover, it is worth noting that for negative feedback, distortion due to the feedback network has an opposite sign to that due to the amplifier.

A more compact and clear representation of the harmonic distortion in a nonlinear amplifier with nonlinear feedback is

$$HD_{2f} = HD_{2fl} + HD_{2fn} \tag{7.36a}$$

$$HD_{3f} = HD_{3fl} + HD_{3fn} + 4HD_{2fl}HD_{2fn} \tag{7.36b}$$

In conclusion, the second and third harmonic distortion terms can be compactly represented by relationships (7.36) which are only a simple function of the second and third harmonic distortion of the whole feedback network evaluated in two particular cases:

- a nonlinear amplifier with linearised feedback network
- a linearised amplifier with nonlinear feedback network.

This consideration can be particularly interesting from a design point of view, since other than allowing us to get more insight into the circuit behaviour and its final performance, permits to evaluate all the harmonic distortion factors through separate calculation (or simulation) of the two couples of terms HD_{2fl}, HD_{3fl} and HD_{2fn}, HD_{3fn} [PP981].

7.2 HARMONIC DISTORTION IN THE FREQUENCY DOMAIN

In the previous paragraphs, both the amplifier and the feedback network were assumed to be frequency independent. This hypothesis is clearly a rough approximation. Transistors have parasitic capacitances which cause the gain and even the nonlinear amplifier coefficients to vary with frequency. Yet, high-gain feedback circuits must be frequency compensated to ensure closed-loop stability, while the feedback network can include reactive (usually capacitive) components. Therefore, the previous expressions can be used with reasonable accuracy only under the hypothesis of low-frequency input signals.

In general, evaluation of harmonic distortion of a *dynamic* system requires complex calculation involving Volterra series or even Wiener series [BR71], [MSE72], [NP73], [WG99]. Nevertheless, under the assumption of low-distortion conditions –which means in practice, that the amplifier output is not saturated and transistors do not leave their active region of operation– we can use the usual small-signal analysis to produce accurate results. Let us start our discussion by considering amplifiers in open-loop configuration.

7.2.1 Open-loop Amplifiers

To render the analysis sufficiently general, we will refer to two-stage amplifiers, that adequately model real amplifiers (the obtained results could then be extended also to multi-stage topologies, as well). Besides, we simplify analysis by separating the effect of nonlinearities of the first and second stage. These two cases are illustrated in Fig. 7.3a and 7.3b. Of course, in real amplifiers both the two phenomena coexist as nonlinearity can contemporarily come from the input and the output sections. Nevertheless, this simplification is instructive and even representative of actual cases. Indeed, the first scheme (Fig. 7.3a) exemplifies a conventional op-amp or a CMOS OTA with a nonlinear output stage. In this event the output section operates in large-signal conditions and its nonlinear behaviour is hence exacerbated. The second scheme (Fig. 7.3b) seems uncommon. Later, we will demonstrate that this case models the high-frequency distortion in single-stage amplifiers. Besides, it can exemplify amplifiers operated under large common-mode input signals, responsible for the generation of nonlinearities in the input stage.

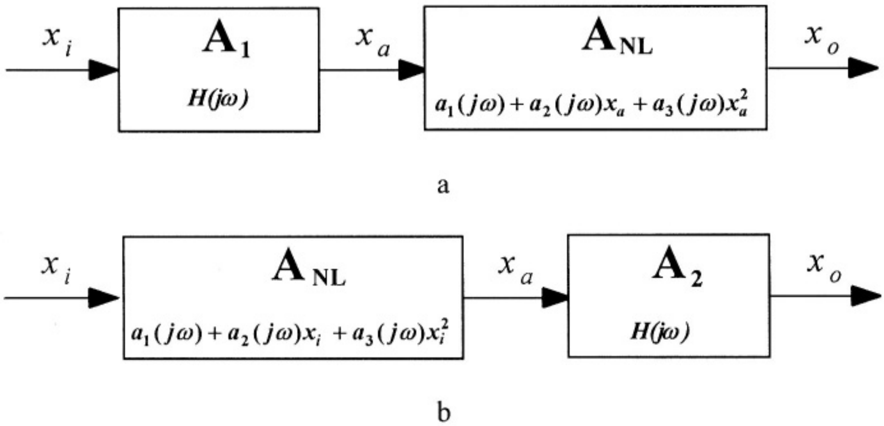

Fig. 7.3a. Amplifier models with: linear input stage and nonlinear output stage a), nonlinear input stage followed by a linear stage b).

Evaluation of harmonic distortion factors for the amplifier schematised in Fig. 7.3a is straightforward. We can use (7.7) and (7.8) after noting that the input signal of the nonlinear block is now $H(j\omega)X_i$. Hence, the distortion factors are

$$HD_2(\omega) \approx \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|}|H(j\omega)|X_i = \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|^2}|X_o(j\omega)| \tag{7.37}$$

$$HD_3(\omega) \approx \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|}|H(j\omega)|^2 X_i^2 = \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|^3}|X_o(j\omega)|^2 \tag{7.38}$$

Harmonic distortion referred to the amplitude of the output signal fundamental component are formally identical to the last equations in (7.7) and (7.8) except that now these expressions must be evaluated at the frequency of the input signal (fundamental). Note that this also holds for X_o, i.e., also the output signal must be calculated at the fundamental frequency. Consequently, when we have to evaluate the frequency behaviour of HD_2 and HD_3, it is easier to refer to their formulations in terms of the input signal X_i.

The above equations give the magnitude of HD_2 and HD_3, as this is the most common information required by designers. However, in their general form these equations can be used to obtain also information on phase distortion. In the following we will consider only the magnitude of distortion factors.

The second basic case considered is that of a nonlinear amplifier followed by a linear stage, as shown in Fig. 7.3b. Assuming, as usual, that the incremental input voltage is a pure sinusoidal tone, $x_i = X_i \cos(\omega t)$, the intermediate output is

$$x_a \approx |b_1(j\omega)|\cos(\omega t) + |b_2(j\omega)|\cos(2\omega t) + |b_3(j\omega)|\cos(3\omega t) \tag{7.39}$$

where coefficients $b_i(j\omega)$ are again given by (7.3)-(7.6) in which $a_i(j\omega)$ have to be used instead of constant values a_i. Then, the output signal is

$$x_o \approx |b_1(j\omega)||H(j\omega)|\cos(\omega t) + |b_2(j\omega)||H(j2\omega)|\cos(2\omega t) + \\ + |b_3(j\omega)||H(j3\omega)|\cos(3\omega t) \tag{7.40}$$

In the above equations the phase contribution of $b_i(j\omega)$ to the x_a components and that of $H(j\omega)$ to x_o has been neglected. Finally, from (7.7) and (7.8) we get

$$HD_2(\omega) \approx \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|}\frac{|H(j2\omega)|}{|H(j\omega)|}X_i = \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|^2}\frac{|H(j2\omega)|}{|H(j\omega)|^2}|X_o(j\omega)| \tag{7.41}$$

$$HD_3(\omega) = \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|}\frac{|H(j3\omega)|}{|H(j\omega)|}X_i^2 = \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|^3}\frac{|H(j3\omega)|}{|H(j\omega)|^3}|X_o(j\omega)|^2 \tag{7.42}$$

Comparing the above expressions with (7.7) we see that the harmonic distortion factors are now multiplied by the ratio of the transfer function magnitudes evaluated at the frequency of the considered harmonics and at the fundamental frequency.

As a particular case, assume that coefficients a_i are constant, and that the transfer function $H(j\omega)$ has a single pole (the pole of the amplifier and also of the loop gain)

$$H(j\omega) = \frac{h}{1 + j\dfrac{\omega}{\omega_c}} \tag{7.43}$$

Accordingly, (7.41) and (7.42) become

$$HD_2(\omega) = \frac{1}{2}\frac{a_2}{a_1}\left|\frac{1+j\omega/\omega_c}{1+j2\omega/\omega_c}\right|X_i = \frac{1}{2}\frac{a_2}{ha_1^2}\frac{\left|1+j\omega/\omega_c\right|^2}{\left|1+j2\omega/\omega_c\right|}\left|X_o(j\omega)\right| \quad (7.44)$$

$$HD_3(\omega) = \frac{1}{4}\frac{a_3}{a_1}\left|\frac{1+j\omega/\omega_c}{1+j3\omega/\omega_c}\right|X_i^2 = \frac{1}{4}\frac{a_3}{h^2a_1^3}\frac{\left|1+j\omega/\omega_c\right|^3}{\left|1+j3\omega/\omega_c\right|}\left|X_o(j\omega)\right|^2 \quad (7.45)$$

The above equations show reduction of the second and third harmonic distortion factors with respect to their low-frequency values. Indeed, at frequencies respectively equal to $\omega_c/2$ and $\omega_c/3$, the asymptotic diagrams of $HD_2(\omega)$ and $HD_3(\omega)$ start to linearly decrease. Then, the distortion factors become constant at the cut-off frequency. This behaviour is qualitatively shown in Fig. 7.4.

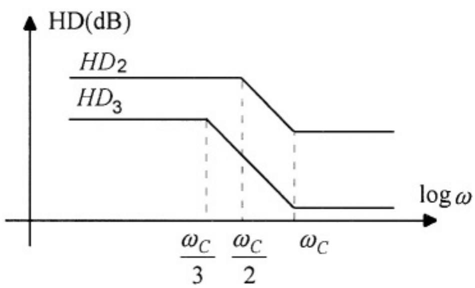

Fig. 7.4. Harmonic distortion factors for the scheme in Fig. 7.3b. Coefficients a_i are assumed constant and ω_c is the pole of $H(j\omega)$.

7.2.2 Closed-loop Amplifiers

Consider now the same feedback amplifier in Fig. 7.2, but where the transfer functions of blocks A_{NL} and f are now frequency dependent. Specifically, let block A_{NL} be characterised by the frequency-dependent nonlinear coefficients $a_1(j\omega)$, $a_2(j\omega)$, and $a_3(j\omega)$, and denote as $F(j\omega)$ the transfer function of linear feedback block f, as schematised in Fig. 7.5.

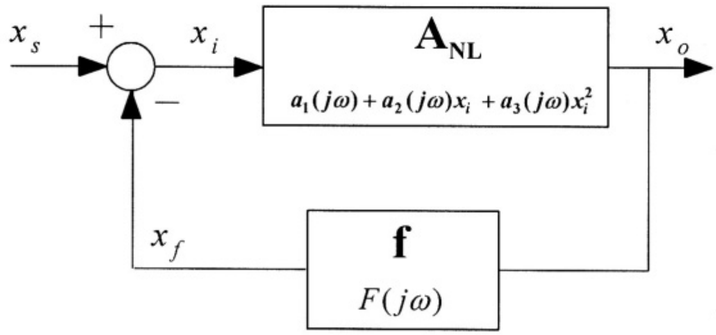

Fig. 7.5. Nonlinear feedback amplifier in the frequency domain.

To derive the distortion factors of the system in Fig. 7.5, we will now develop an intuitive method which requires simple algebraic manipulations. The approach leads to expressions of distortion factors that are a direct extension of those in (7.16) (7.17) found at low frequency.

As usual, we assume a sinusoidal input tone $x_s = X_s\cos(\omega t)$ and that it is possible to write the output signal as a power series of the source signal

$$x_o \approx |a_1'(j\omega)|X_s\cos(\omega t+\angle a_1'(j\omega))+\frac{1}{2}|a_2'(j\omega)|X_s^2\cos(2\omega t+\angle a_2'(j\omega))$$
$$+\frac{1}{4}|a_3'(j\omega)|X_s^3\cos(3\omega t+\angle a_3'(j\omega)) \qquad (7.46)$$

The problem is to find the expression of the closed-loop nonlinear coefficients $a'_i(j\omega)$.

The first coefficient $a'_1(j\omega)$, which is responsible for the linear behaviour, can be simply found. It is equal to the forward-path transfer function divided by 1 plus the loop-gain transfer function, $T(j\omega)$

$$a'_1(j\omega)=\frac{a_1(j\omega)}{1+a_1(j\omega)F(j\omega)}=\frac{a_1(j\omega)}{1+T(j\omega)} \qquad (7.47)$$

Equation (7.47) implies computation of $T(j\omega)$ and $a_1(j\omega)$ at the frequency of the input tone (i.e., the fundamental frequency).

To evaluate the higher-order coefficients we have to follow a simple, but not trivial reasoning. Concentrate our attention to derive the second harmonic component at the output. It is produced by the nonlinear block when a signal at the fundamental frequency is presented to its input. Now observe that the second harmonics is proportional to x_i^2. If the circuit is perfectly linear (i.e., $a_2(j\omega) = a_3(j\omega) = 0$), x_i would be equal to $x_s / |1 + T(j\omega)|$. Therefore, the nonlinear block produces a second harmonic component with amplitude equal to $\frac{1}{2}|a_2(j\omega)| \frac{1}{|1 + T(j\omega)|^2} X_s^2$. This component can be viewed as a spurious signal injected at the output of the nonlinear block, as depicted in Fig. 7.6. It is subsequently processed by the feedback loop and appears at the output terminal decreased by the loop gain but evaluated at the *frequency of the harmonic considered,* i.e., 2ω.

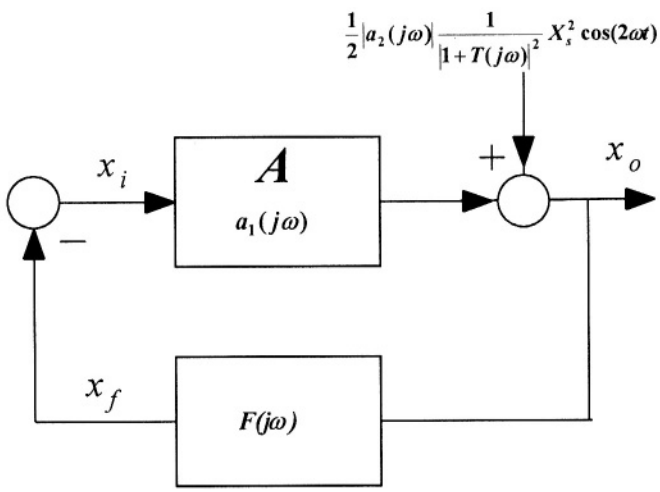

Fig. 7.6. Equivalent representation to evaluate second harmonic distortion in the frequency domain.

From the above discussion it follows that the nonlinear term $a'_2(j\omega)$ is equal to

$$a'_2(j\omega) = \frac{1}{[1 + T(j\omega)]^2} \frac{1}{1 + T(j2\omega)} a_2(j\omega) \qquad (7.48)$$

The nonlinear coefficient $a'_3(j\omega)$ can be evaluated by following a similar procedure. Neglecting the contribute due to $a_2(j\omega)$ we get

$$a'_3(j\omega) = \frac{1}{[1+T(j\omega)]^3} \frac{1}{1+T(j3\omega)} a_3(j\omega) \tag{7.49a}$$

Taking into account also the effect of $a_2(j\omega)$ an expression similar to (7.15) can be deduced. At this purpose, we consider the schematisation depicted in Fig. 7.7 which leads to

$$a'_3(j\omega) \approx \frac{1}{[1+T(j\omega)]^3} \frac{1}{1+T(j3\omega)} a_3(j\omega)\left[1 - \frac{2a_2^2(j\omega)}{a_1(j\omega)a_3(j\omega)}\right] \tag{7.49b}$$

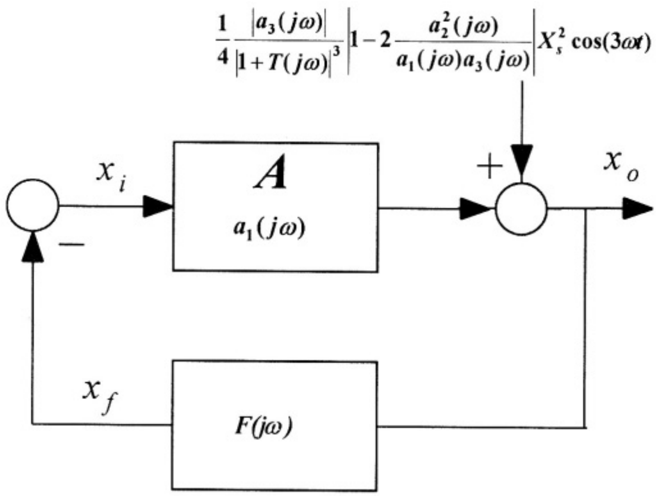

Fig. 7.7. Equivalent representation to evaluate complete third harmonic distortion in the frequency domain.

Substituting (7.47), (7.48) and (7.49b) into (7) and (8) we get

$$HD_{2f}(\omega) = \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|}\frac{1}{|1+T(j\omega)||1+T(j2\omega)|}X_s =$$

$$= \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|^2}\frac{1}{|1+T(j2\omega)|}|X_o(j\omega)| \tag{7.50}$$

$$HD_{3f}(\omega) = \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|}\frac{1}{|1+T(j\omega)|^2|1+T(j3\omega)|}\left|1-2\frac{a_2^2(j\omega)}{a_1(j\omega)a_3(j\omega)}\right|X_s^2$$

$$= \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|^3}\frac{1}{|1+T(j3\omega)|}\left|1-2\frac{a_2^2(j\omega)}{a_1(j\omega)a_3(j\omega)}\right||X_o(j\omega)|^2 \tag{7.51}$$

Of course, the above equations adhere with (7.16), (7.17) and (7.19a) found in the case of frequency-independent loop gain, or that is the same, for low-frequency input signals, in the present case, distortion of a feedback network in terms of the output fundamental is reduced of a quantity still equal to the return ratio but evaluated at the *considered* harmonic.

It is useful to extend these results to a more general model in which we put the nonlinear block between two linear blocks in the forward path, as shown in Fig. 7.8a. This system includes as particular cases the closed-loop version of both occurrences, depicted in Fig. 7.3a and 7.3b, in which distortion appears after or before a linear stage.

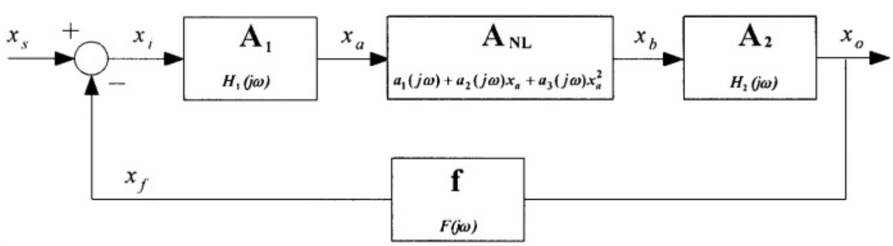

Fig. 7.8. A general model of closed-loop nonlinear amplifier for evaluation of harmonic distortion in the frequency domain.

To obtain distortion factors of the system in Fig. 7.8, we can follow the same procedure described above. Let us first evaluate the nonlinear coefficients that relate x_o to x_s. The first-order coefficient is

$$a'_1(j\omega) = \frac{H_1(j\omega)H_2(j\omega)}{1+T(j\omega)}a_1(j\omega) \tag{7.52}$$

where $T(j\omega) = a_1(j\omega)F(j\omega)H_1(j\omega)H_2(j\omega)$.

To obtain the second-order coefficient it is convenient to refer to Fig. 7.9, which illustrates the second-order component injected at the output of the nonlinear block

$$a'_2(j\omega) = \frac{H_1(j\omega)^2}{[1+T(j\omega)]^2}\frac{H_2(j2\omega)}{1+T(j2\omega)}a_2(j\omega) \tag{7.53}$$

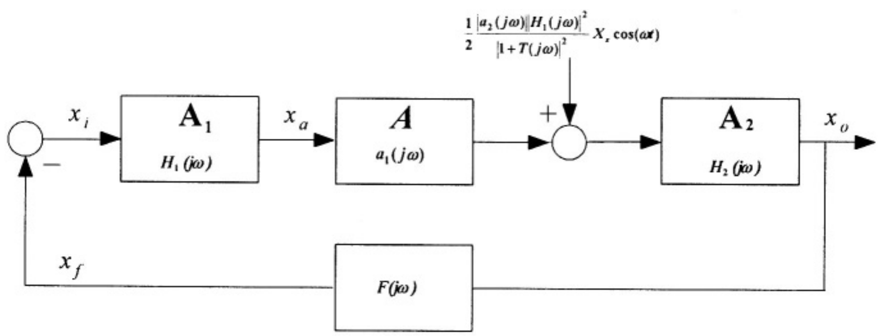

Fig. 7.9. Equivalent representation to evaluate the second harmonic distortion for system in Fig. 7.8.

A similar procedure can be applied to the third-order coefficient, yielding

$$a'_3(j\omega) = \left[\frac{H_1(j\omega)}{1+T(j\omega)}\right]^3 \frac{H_2(j3\omega)}{1+T(j3\omega)}a_3\left[1-\frac{2a_2^2(j\omega)}{a_3(j\omega)a_1(j\omega)}\right] \tag{7.54}$$

Then, the harmonic distortion factors are expressed by

$$\begin{aligned} HD_{2f}(\omega) &= \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|}\frac{|H_1(j\omega)||H_2(j2\omega)|}{|1+T(j\omega)||1+T(j2\omega)||H_2(j\omega)|}X_s = \\ &= \frac{1}{2}\frac{|a_2(j\omega)|}{|a_1(j\omega)|^2}\frac{|H_2(j2\omega)|}{|1+T(j2\omega)||H_2(j\omega)|^2}|X_o(j\omega)| \end{aligned} \tag{7.55}$$

$$HD_{3f}(\omega) = \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|}\frac{|H_1(j\omega)|^2|H_2(j3\omega)|}{|1+T(j\omega)|^2|1+T(j3\omega)||H_2(j\omega)|}\cdot\left|1-2\frac{a_2^2(j\omega)}{a_3(j\omega)a_1(j\omega)}\right|X_s^2$$

$$= \frac{1}{4}\frac{|a_3(j\omega)|}{|a_1(j\omega)|^3}\frac{|H_2(j3\omega)|}{|1+T(j3\omega)||H_2(j\omega)|^3}\cdot\left|1-2\frac{a_2^2(j\omega)}{a_3(j\omega)a_1(j\omega)}\right||X_o(j\omega)|^2 \tag{7.56}$$

7.3 HARMONIC DISTORTION AND COMPENSATION

In this paragraph we will study the effect of the different types of frequency compensation on harmonic distortion. To this end, we will first apply the above results to two-stage amplifiers and then a typical single-stage amplifier will be considered. Dominant-pole and Miller techniques for a two-stage amplifier are treated in sections 7.3.1 and 7.3.2, respectively. Under the assumption that the second stage is the principal responsible for nonlinear behaviour, we will demonstrate the better linearity performance of Miller-compensated amplifiers. Linearity performance of a single-stage architecture with dominant-pole compensation will be treated in section 7.3.3.

Linear and, unless specified, frequency-independent feedback is thorough considered for simplicity.

7.3.1 Two-stage Amplifier with Dominant-Pole Compensation

The analysis carried out in the previous paragraph can be now directly applied to two-stage amplifiers compensated with the dominant-pole technique.

We can use the same model in Fig. 7.8, and assume $H_2(j\omega) = 1$ and $H_1(j\omega)$ to be a single-pole transfer function given by (7.43) and here reported again

$$H_1(j\omega) = \frac{h}{1 + j\omega/\omega_c} \tag{7.57}$$

This means that the nonlinearity is caused by the second stage. Assume also for simplicity the nonlinear coefficients a_i being independent of frequency.

From (7.50) and (7.51), being $T_o = hfa_1$ and $\omega_{GBW} = (1+T_o)\omega_c$, we get immediately

$$HD_{2f}(\omega) = \frac{1}{2}\frac{a_2}{a_1}\frac{h}{(1+T_o)^2}\frac{\left|1+j2\frac{\omega}{\omega_c}\right|}{\left|1+j\frac{2\omega}{\omega_{GBW}}\right|\left|1+j\frac{\omega}{\omega_{GBW}}\right|}X_s$$

$$= \frac{1}{2}\frac{a_2}{a_1^2}\frac{1}{(1+T_o)}\frac{\left|1+j2\frac{\omega}{\omega_c}\right|}{\left|1+j\frac{2\omega}{\omega_{GBW}}\right|}\left|X_o(j\omega)\right| \tag{7.58}$$

$$HD_{3f}(\omega) = \frac{1}{4}\frac{a_3}{a_1^3}\left(1-2\frac{a_2^2}{a_3 a_1}\right)\frac{h^2}{(1+T_o)^3}\frac{\left|1+j3\frac{\omega}{\omega_c}\right|}{\left|1+j\frac{\omega}{\omega_{GBW}}\right|\left|1+j\frac{3\omega}{\omega_{GBW}}\right|}X_s \tag{7.59}$$

$$= \frac{1}{4}\frac{a_3}{a_1^3}\left(1-2\frac{a_2^2}{a_3 a_1}\right)\frac{1}{(1+T_o)}\frac{\left|1+j3\frac{\omega}{\omega_c}\right|}{\left|1+j\frac{3\omega}{\omega_{GBW}}\right|}\left|X_o(j\omega)\right|^2$$

Second- and third-order harmonic distortion factors start to *linearly* increase (from their low-frequency values) at a frequency equal to $\omega_c/2$ and $\omega_c/3$, respectively. Moreover, they become constant at frequencies equal to $\omega_{GBW}/2$ and $\omega_{GBW}/3$, respectively. At ω_{GBW} they begin to decrease.

A final observation concerns the distortion caused by the first amplifier stage. Nonlinear contributions of the input stage are reduced by the loop gain at low frequencies, and by the compensation capacitor at high frequencies (compensation tends to shunt the output of the first stage). Therefore, assuming the output stage as a principal source of nonlinearity is very well justified both for low and high frequencies. We will show in the following that this assumption is inadequate for Miller-compensated amplifiers.

7.3.2 Two-stage Amplifier with Miller Compensation

Another important case study is the evaluation of distortion for a two-stage Miller-compensated amplifier. Let us first analyse the open-loop amplifier in Fig. 7.10a, in which the second stage is nonlinear. In the figure, R and v_1 are the output resistance and the output voltage of the first stage, whose transconductance is G_m. The second stage, instead, is modeled by a voltage-controlled voltage source, A_{NL}, to preserve simplicity. To this end, we can also model the first stage with its Thévenin equivalent. The open-loop output voltage is then expressed by

$$v_{out} = A_{NL}(v_1) = -\left(a_1 + a_2 v_1 + a_3 v_1^2\right)v_1 \tag{7.60}$$

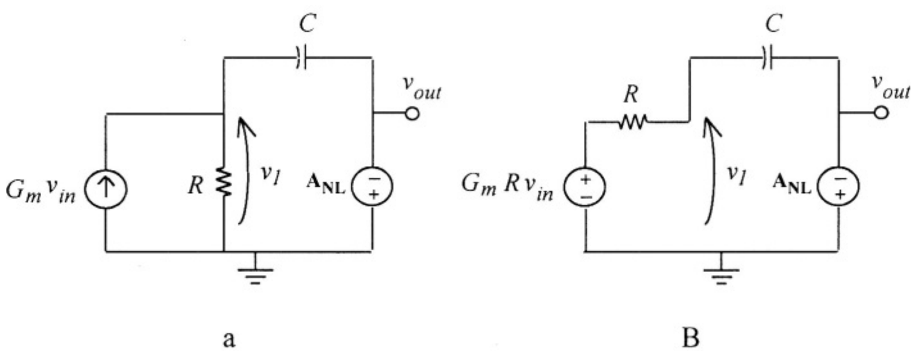

Fig. 7.10. Models of a Miller-compensated two-stage amplifier with a nonlinear output stage: Norton equivalent input stage a), Thévenin equivalent input stage.

The return ratio and the asymptotic gain of the amplifier in Fig. 7.10 are

$$T(j\omega) = \frac{j\omega RC}{1 + j\omega RC} a_1 \tag{7.61}$$

$$G_\infty(j\omega) = -\frac{1}{j\omega RC} \tag{7.62}$$

Given the Miller effect, we can consider the pole as being placed at the output of the first stage. Thus to analyse the circuit, we can use an equivalent block diagram similar to the one in Fig. 7.8 and depicted in Fig. 7.11 in which the nonlinear amplifier A_{NL} is characterised by the same nonlinear coefficients in (7.60).

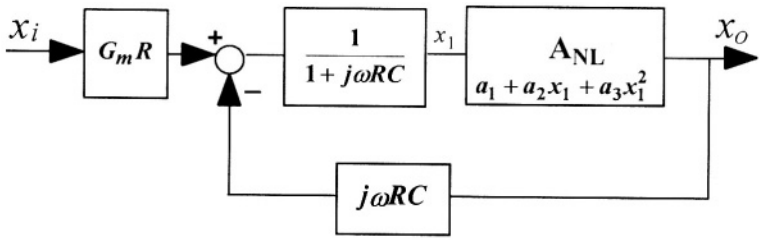

Fig. 7.11. Equivalent block diagram of the amplifier in Fig. 7.10b.

By comparing the general model in Fig. 7,8 with that in Fig. 7.11 we can utilise (7.52)-(7.54) to obtain the expression of the nonlinear closed-loop coefficients, where $H_2(j\omega) = 1$, $1/f = |G_\infty|$ and

$$H_1(j\omega) = \frac{G_\infty(j\omega)T(j\omega)}{a_1} = \frac{1}{1 + j\omega RC} \tag{7.63}$$

In addition, the closed-loop gain results

$$A_f(j\omega) = \frac{G_m R a_1}{1 + j\omega RC(1 + a_1)} \tag{7.64}$$

which, despite the different sign (inessential in evaluating distortion), equals the transfer function obtained by a direct inspection of the circuit in Fig. 7.10. Then, from relationship (7.52)-(7.54) we get the equivalent nonlinear coefficients $a'_1(j\omega)$, $a'_2(j\omega)$, and $a'_3(j\omega)$ which relate x_o to x_i in Fig. 7.11

$$a'_1(j\omega) = \frac{1}{1 + j\omega RC(1 + a_1)} a_1 \tag{7.65}$$

$$a'_2(j\omega) = \frac{1 + j2\omega RC}{[1 + j\omega RC(1 + a_1)]^2 [1 + j2\omega RC(1 + a_1)]} a_2 \tag{7.66}$$

$$a'_3(j\omega) = \frac{1 + j3\omega RC}{[1 + j\omega RC(1 + a_1)]^3 [1 + j3\omega RC(1 + a_1)]} a_3 \left(1 - 2\frac{a_2^2}{a_1 a_3}\right) \tag{7.67}$$

The closed-loop Miller-compensated amplifier can then be modeled as depicted in Fig. 7.12, where the amplifier studied above is closed in a loop with feedback block f. Note that to further simplify the scheme, Fig. 6.12b includes the new nonlinear block A'_{NL}, with its nonlinear coefficients $a'_1(j\omega)$, $a'_2(j\omega)$, and$a'_3(j\omega)$ defined above. Moreover, for conformity with the notation used in the previous section, we define the gain of the first block, *h*, as equal to G_mR.

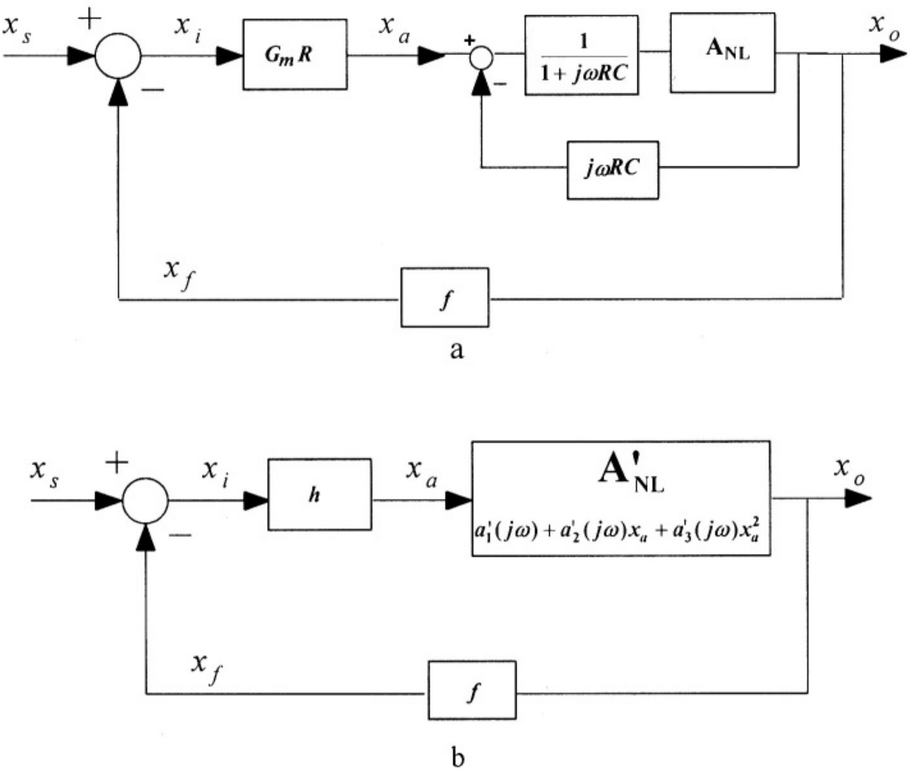

Fig. 7.12. Two-stage Miller-compensated closed-loop amplifier models.

Gain h in Fig. 7.12b is equal to G_mR and coefficients a'_i are defined in (7.65-(7.67).

From relationships (7.50)-(7.51a), and given that $T_o = hfa_1$, we get the second and third harmonic distortion factors

$$HD_{2f}(\omega) = \frac{1}{2}\frac{a_2}{a_1}\frac{h}{(1+T_o)^2}\frac{|1+j2\omega RC|}{\left|1+j2\omega RC\frac{1+a_1}{1+T_o}\right|\left|1+jRC\frac{1+a_1}{1+T_o}\right|}X_s =$$

$$= \frac{1}{2}\frac{a_2}{a_1^2}\frac{1}{1+T_o}\frac{|1+j2\omega RC|}{\left|1+j2\omega RC\frac{1+a_1}{1+T_o}\right|}|X_o(j\omega)| \tag{7.68}$$

$$HD_{3f}(\omega) \approx \frac{1}{4}\frac{a_3}{a_1}\frac{h^2}{(1+T_o)^3}\left(1-2\frac{a_2^2}{a_1a_3}\right)\frac{|1+j3\omega RC|}{\left|1+j3\omega RC\frac{1+a_1}{1+T_o}\right|\left|1+j\omega RC\frac{1+a_1}{1+T_o}\right|^2}X_s^{\,2} =$$

$$= \frac{1}{4}\frac{a_3}{a_1^3}\frac{1}{1+T_o}\left(1-2\frac{a_2^2}{a_1a_3}\right)\frac{|1+j3\omega RC|}{\left|1+j3\omega RC\frac{1+a_1}{1+T_o}\right|}|X_o(j\omega)|^2 \tag{7.69}$$

where in (7.69) we have only considered the dominant terms.

To better compare the above results with those obtained in the case of dominant pole compensation we must express (7.68)-(7.69) in terms of ω_c, that is now equal to $1/[RC(1+a_1)]$ and ω_{GBW} equal to $(1+T_o)\omega_c$

$$HD_{2f}(\omega) = \frac{1}{2}\frac{a_2}{a_1}\frac{h}{(1+T_o)^2}\frac{\left|1+j2\frac{\omega}{(1+a_1)\omega_c}\right|}{\left|1+j2\frac{\omega}{\omega_{GBW}}\right|\left|1+j\frac{\omega}{\omega_{GBW}}\right|}X_s =$$

$$= \frac{1}{2}\frac{a_2}{a_1^2}\frac{1}{1+T_o}\frac{\left|1+j2\frac{\omega}{(1+a_1)\omega_c}\right|}{\left|1+j2\frac{\omega}{\omega_{GBW}}\right|}|X_o(j\omega)| \tag{7.70}$$

$$HD_{3f}(\omega) \approx \frac{1}{4}\frac{a_3}{a_1}\frac{h^2\left(1-2\dfrac{a_2^2}{a_1a_3}\right)}{(1+T_o)^3}\frac{\left|1+j3\dfrac{\omega}{(1+a_1)\omega_c}\right|}{\left|1+j3\dfrac{\omega}{\omega_{GBW}}\right|\left|1+j\dfrac{\omega}{\omega_{GBW}}\right|^2}X_s^{\,2} =$$

$$= \frac{1}{4}\frac{a_3}{a_1^3}\frac{1-2\dfrac{a_2^2}{a_1a_3}}{1+T_o}\frac{\left|1+j3\dfrac{\omega}{(1+a_1)\omega_c}\right|}{\left|1+j3\dfrac{\omega}{\omega_{GBW}}\right|}\left|X_o(j\omega)\right|^2 \qquad (7.71)$$

Starting from their low-frequency values, second- and third-order harmonic distortion factors linearly increase at a frequency equal to $(1+a_1)\omega_c/2$ and $(1+a_1)\omega_c/3$, respectively. Compared to dominant-pole compensation, we see that the frequency band where distortion factors remain equal to their low-frequency values is greater in the Miller-compensated amplifier by a factor equal to $1+a_1$.

Equations (7.70) and (7.71) also predict that HD_{2f} and HD_{3f} become constant at frequencies equal to $\omega_{GBW}/2$ and $\omega_{GBW}/3$, respectively. At ω_{GBW} they begin to decrease. This behaviour was already found appropriate in two-stage amplifiers compensated with a dominant pole. In contrast, when using Miller compensation it is unrealistic. Indeed, the local feedback operated by the Miller capacitor causes coefficients $a_i'(j\omega)$ to decrease with frequency. At high frequencies, distortion of the first stage becomes dominant and a nonlinear model of the first stage should then be included to accurately predict harmonic distortion.

The use of nonlinear models for both the first and second stage considerably complicates distortion evaluation. However, since the two distortion mechanisms are dominant over different frequency ranges (distortion due to the input stage is effective at high frequencies, whilst distortion due to the output stage is dominant at low frequencies) we can separately study the two cases with our distortion models[4]. We shall not use this approach now, because it can be shown that fairly good approximation for distortion factors valid up to the gain-bandwidth product is found simply

[4] An example of how to treat distortion coming from two cascaded stages is described in the next section, 7.3.2.

by eliminating the poles in (7.70) and (7.71) respectively at ω_{GBW}, $\omega_{GBW}/2$ and at ω_{GBW}, $\omega_{GBW}/3$.

As a result, HD_{2f} and HD_{3f} for a two-stage amplifier compensated with Miller technique are expressed by

$$HD_{2f}(\omega) \approx \frac{1}{2}\frac{a_2}{a_1}\frac{h}{(1+T_o)^2}\left|1+j2\frac{\omega}{(1+a_1)\omega_c}\right|X_s = \\ = \frac{1}{2}\frac{a_2}{a_1^2}\frac{1}{1+T_o}\left|1+j2\frac{\omega}{(1+a_1)\omega_c}\right|\left|1+j\frac{\omega}{\omega_{GBW}}\right|\left|X_o(j\omega)\right| \tag{7.72}$$

$$HD_{3f}(\omega) \approx \frac{1}{4}\frac{a_3}{a_1}\frac{h^2\left(1-2\frac{a_2^2}{a_1 a_3}\right)}{(1+T_o)^3}\left|1+j3\frac{\omega}{(1+a_1)\omega_c}\right|X_s^2 = \\ = \frac{1}{4}\frac{a_3}{a_1^3}\frac{1-2\frac{a_2^2}{a_1 a_3}}{1+T_o}\left|1+j3\frac{\omega}{(1+a_1)\omega_c}\right|\left|1+j\frac{\omega}{\omega_{GBW}}\right|^2\left|X_o(j\omega)\right|^2 \tag{7.73}$$

To qualitatively illustrate the improvement in linearity of Miller compensation over dominant-pole compensation, Figure 7.13 shows the HD_{2f} achieved in both cases. A similar plot can be drawn for HD_{3f}

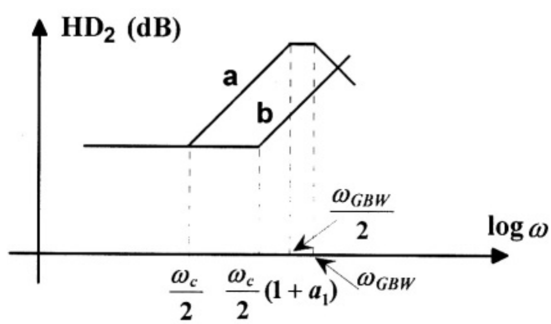

Fig. 7.12. Typical behaviour of second-order distortion factors for two-stage amplifiers with dominant-pole compensation (curve a) and Miller compensation (curve b).

7.3.3 Single-stage Amplifiers

The last case we shall study is that of the single-stage amplifiers. These architectures are frequently employed in IC applications (for instance in switched-capacitor circuits) for their high-frequency performance. Indeed, a single-stage amplifier exhibits only an (output) high-resistance node. Moreover, this output node often exploits cascoding, allowing a voltage gain similar to that of two-stage amplifiers to be achieved. Of course, these amplifiers are used in closed-loop configurations and, due to the internal structure, output dominant-pole compensation is invariably utilised.

The small-signal model of a (open-loop) single-stage amplifier is illustrated in Fig. 7.13, in which *C* is the output compensation capacitor.

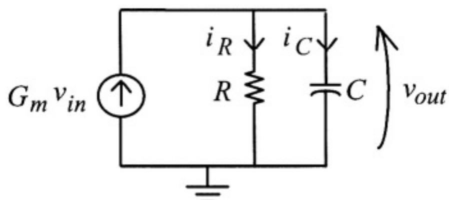

Fig. 7.13. Small-signal model of a single-stage amplifier.

In general, there are two sources of harmonic distortion in such amplifiers. The first is due to the nonlinear V-I conversion accomplished by the input transconductance stage. The second is due to the nonlinear I-V characteristic exhibited by the output devices.

Let us first analyse the effect on linearity of the nonlinear output resistance. Observe that this case does not fall into the category of any of those already studied because both pole and distortion are generated at the same circuit node (i.e., the output) by the same nonlinear element. Hence a specific analysis must be performed.

For easy calculation express the input signal as $v_{in} = V_M e^{j\omega t}$. Moreover, it is better to characterise the nonlinear resistance in terms of (nonlinear) conductance

$$i_R = \frac{1}{R}\left(1 + g_{2N} v_{out} + g_{3N} v_{out}^2\right) v_{out} \tag{7.72}$$

where g_{2N} and g_{3N} are nonlinear coefficients normalised to the linear part of the output conductance $1/R$. These cause harmonic distortion components to appear in the output voltage, according to

$$v_{out} = b_1(j\omega)\cdot V_M e^{j\omega t} + b_2(j\omega)\cdot V_M^2 e^{j2\omega t} + b_3(j\omega)\cdot V_M^3 e^{j3\omega t} \qquad (7.73)$$

in which only the first three terms are taken. Then, the current through the capacitor is

$$i_C = C\frac{dv_{out}}{dt} = \qquad (7.74)$$

$$= C\left[j\omega b_1(j\omega)\cdot V_M e^{j\omega t} + j2\omega b_2(j\omega)\cdot V_M^2 e^{j2\omega t} + j3\omega b_3(j\omega)\cdot V_M^3 e^{j3\omega t}\right]$$

From the KCL at the output node

$$G_m v_{in} = i_R + i_C \qquad (7.75)$$

using (7.74) and the current through the nonlinear resistor found by substituting (7.73) in (7.72), and equating all the harmonic components with the same frequency, we can derive the expression of coefficients $b_1(j\omega)$, $b_2(j\omega)$, and $b_3(j\omega)$. Thus, considering only the dominant terms we get

$$b_1(j\omega) = \frac{G_m R}{1 + j\omega RC} \qquad (7.76)$$

$$b_2(j\omega) = -\frac{g_{2N}(G_m R)^2}{(1 + j\omega RC)^2(1 + j2\omega RC)} \qquad (7.77)$$

$$b_3(j\omega) = -\frac{(G_m R)^3}{(1 + j\omega RC)^3(1 + j3\omega RC)}\left(g_{3N} - \frac{2g_{2N}^2}{1 + j2\omega RC}\right) \qquad (7.78)$$

Normalising the second and third coefficient to $b_1(j\omega)$, and given that $\omega_c = 1/RC$, we get

$$b_{2N}(j\omega) = \frac{b_2(j\omega)}{b_1(j\omega)} = -\frac{g_{2N}G_m R}{\left(1 + j\dfrac{\omega}{\omega_c}\right)\left(1 + j\dfrac{2\omega}{\omega_c}\right)} \qquad (7.79)$$

$$b_{3N}(j\omega)=\frac{b_3(j\omega)}{b_1(j\omega)}=-\frac{(G_mR)^2}{\left(1+j\frac{\omega}{\omega_c}\right)^2\left(1+j\frac{3\omega}{\omega_c}\right)}\left(g_{3N}-\frac{2g_{2N}^2}{1+j\frac{2\omega}{\omega_c}}\right) \quad (7.80)$$

Hence, the feedback circuit can then be schematised by the block diagram in Fig. 7.14, where the blocks inside the shadowed area represent the linear and nonlinear contributes of the RC output node, with the nonlinear coefficients given by

$$b'_{2N}(j\omega)=-\frac{g_{2N}}{\left(1+j\frac{2\omega}{\omega_c}\right)} \quad (7.81)$$

$$b'_{3N}(j\omega)=-\frac{1}{\left(1+j\frac{3\omega}{\omega_c}\right)}\left(g_{3N}-\frac{2g_{2N}^2}{1+j\frac{2\omega}{\omega_c}}\right) \quad (7.82)$$

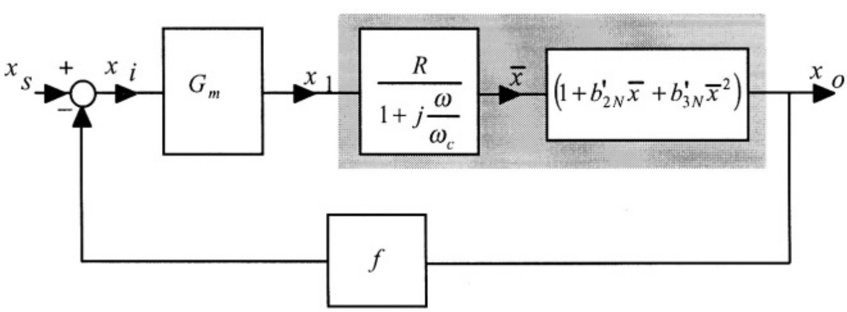

Fig. 7.14. Model of a single-stage amplifier with a nonlinear output resistance.

To evaluate the closed-loop harmonic distortion factors we can employ the results found at the end of section 7.2.2. After applying (7.55) and (7.56) we get the following equations in which $T_o = fG_mR$ and $\omega_{GBW} = (1+T_o)\omega_c$ (expressions in terms of X_s only are reported for compactness).

$$
\begin{aligned}
HD_{2f}^{R}(\omega) &= \frac{1}{2}\frac{G_m b_{2N}}{\left|1+T(j\omega)\right|\left|1+T(j2\omega)\right|}X_s = \\
&= \frac{1}{2}\frac{G_m R}{\left(1+T_o\right)^2}\frac{g_{2N}}{\left|1+j\dfrac{\omega}{\omega_{GBW}}\right|\left|1+j\dfrac{2\omega}{\omega_{GBW}}\right|}X_s
\end{aligned}
\tag{7.83}
$$

$$
\begin{aligned}
HD_{3f}^{R}(\omega) &= \frac{1}{4}\frac{G_m^2 b_{3N}}{\left|1+T(j\omega)\right|^2\left|1+T(j2\omega)\right|}X_s^2 = \\
&= \frac{1}{4}\frac{\left(G_m R\right)^2}{\left(1+T_o\right)^3}\frac{g_{3N}-\dfrac{2g_{2N}^2}{1+j\dfrac{2\omega}{\omega_c}}}{\left|1+j\dfrac{\omega}{\omega_{GBW}}\right|^2\left|1+j\dfrac{3\omega}{\omega_{GBW}}\right|}X_s^2
\end{aligned}
\tag{7.84}
$$

Distortion due to the nonlinear output conductance is effective at low frequencies. Indeed, so long as the loop gain is high, signal x_i (the *error* signal) is small, and distortion is mainly due to nonlinearities arising in the output resistance R which is operated under large-signal conditions. For increasing frequencies the compensation capacitor shunts the output impedance to ground thereby reducing the weight of nonlinearities due to the output resistance. Moreover, signal x_i increases (due to the reduction in the loop gain) and the nonlinear effects of the input transconductance become more pronounced. Thus at high frequencies the amplifier is more adequately modeled by the block diagram in Fig. 7.15, which includes normalised nonlinear coefficients of the input transconductance a_{2N} and a_{3N}, and assumes the output resistance to be linear.

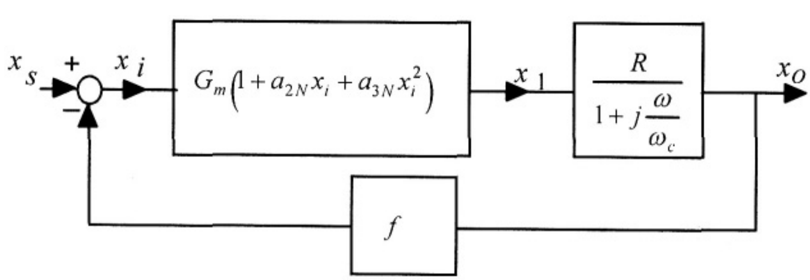

Fig. 7.15. Model of a single-stage amplifier with a nonlinear input transconductance.

This scheme is equivalent to the one analysed in Fig. 7.8 by properly updating the block transfer functions. Hence, utilising (7.55) and (7.56) we get

$$HD_{2f}^{Gm}(\omega)=\frac{1}{2}\frac{a_{2N}}{\left|1+T(j\omega)\right|\left|1+T(j2\omega)\right|}\frac{R}{\left|1+j\frac{\omega}{\omega_c}\right|}\frac{R}{\left|1+j\frac{2\omega}{\omega_c}\right|}X_s= \tag{7.85}$$

$$=\frac{1}{2}\frac{a_{2N}}{\left(1+T_o\right)^2}\frac{\left|1+j\frac{\omega}{\omega_c}\right|^2}{\left|1+j\frac{\omega}{\omega_{GBW}}\right|\left|1+j\frac{2\omega}{\omega_{GBW}}\right|}X_s$$

$$HD_{3f}^{Gm}(\omega)=\frac{1}{4}\frac{a_{3N}\left(1-2\frac{a_{2N}^2}{a_{3N}}\right)}{\left|1+T(j\omega)\right|^2\left|1+T(j2\omega)\right|}\frac{R}{\left|1+j\frac{\omega}{\omega_c}\right|}\frac{R}{\left|1+j\frac{3\omega}{\omega_c}\right|}X_s^2= \tag{7.86}$$

$$=\frac{1}{4}\frac{a_{3N}-2a_{2N}^2}{\left(1+T_o\right)^3}\frac{\left|1+j\frac{\omega}{\omega_c}\right|^3}{\left|1+j\frac{\omega}{\omega_{GBW}}\right|^2\left|1+j\frac{3\omega}{\omega_{GBW}}\right|}X_s^2$$

Both the above distortion factors increase for frequencies higher than the amplifier pole. As a consequence, their effects can be significant at high frequencies.

To qualitatively compare the effects on output distortion due to the output resistance and the input transconductance, let us consider the plots in Fig. 7.16. They illustrate the typical behaviour of second harmonic distortion factors due to the nonlinear output resistance, HD_{2f}^{R}, and due to the input transconductance, HD_{2f}^{Gm}. The frequency determining which contribution is dominant is located between ω_c and $\omega_{GBW}/2$ and is close to ω_c if $HD_{2f}^{R}(0)$ approaches $HD_{2f}^{Gm}(0)$.

Similar plots can also be deduced for the third harmonic distortion factors.

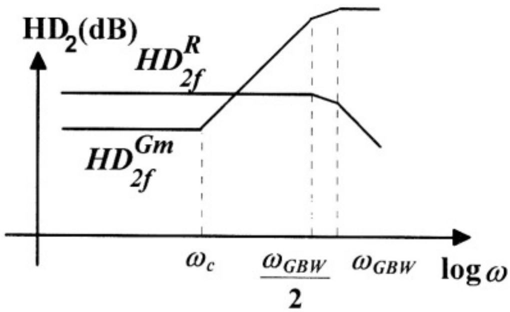

Fig. 7.16. Typical plots of second harmonic distortion factors due to the nonlinear output resistance, HD_{2f}^{R}, and the input transconductance, HD_{2f}^{Gm}.

As a final analysis step, we consider the two distortion mechanisms together in the same block scheme as depicted in Fig. 7.17

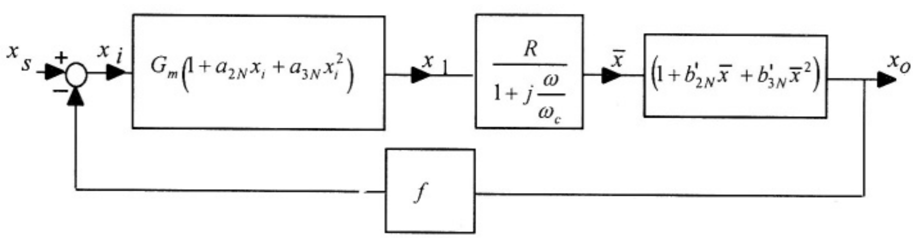

Fig. 7.17. Model of a single-stage amplifier with both nonlinear input transconductance and output resistance.

The exact resolution of this system is difficult, but can fortunately be avoided by considering that the two distortion mechanisms are dominant over different frequency ranges, as previously stated. Consequently, expressions of complete distortion factors HD_{2f} and HD_{3f} which provide asymptotic approximation can be found by combining (7.83) with (7.85) and (7.84) with (7.86)

$$HD_{2f}(\omega)=\frac{1}{2}\frac{G_m R}{(1+T_o)^2}\frac{\left|g_{2N}+\frac{a_{2N}}{G_m R}\left(1+j\frac{\omega}{\omega_c}\right)^2\right|}{\left|1+j\frac{\omega}{\omega_{GBW}}\right|\left|1+j\frac{2\omega}{\omega_{GBW}}\right|}X_s \tag{7.87}$$

$$HD_{3f}(\omega)=\frac{1}{4}\frac{(G_m R)^2}{(1+T_o)^3}\frac{\left|g_{3N}-\frac{2g_{2N}^2}{1+j\frac{2\omega}{\omega_c}}+\frac{a_{3N}-2a_{2N}^2}{(G_m R)^2}\left(1+j\frac{\omega}{\omega_c}\right)^3\right|}{\left|1+j\frac{\omega}{\omega_{GBW}}\right|^2\left|1+j\frac{3\omega}{\omega_{GBW}}\right|}X_s \tag{7.88}$$

The above relationships have simply been obtained by algebraically adding, before taking their modules, HD_{2f}^{R} with HD_{2f}^{Gm} and HD_{3f}^{R} with HD_{3f}^{Gm}.

7.4 AN ALTERNATIVE FREQUENCY ANALYSIS

In this paragraph we describe a simple analytical procedure to calculate the closed-loop harmonic distortion factors in the frequency domain, already found in section 7.2.2 through an euristic demonstration, and used in this chapter. Refer again to Fig. 7.5 and express the source signal as

$$x_s = X_s e^{j\omega t} \tag{7.89}$$

Due to the nonlinear block in the direct path the output signal will include harmonic components. Assume it is given by

$$x_o = a_1'(j\omega)X_s e^{j\omega t} + a_2'(j\omega)X_s^2 e^{j2\omega t} + a_3'(j\omega)X_s^3 e^{j3\omega t} \tag{7.90}$$

where coefficients $a_1'(j\omega)$, $a_2'(j\omega)$, and $a_3'(j\omega)$ have to be determined.

The error signal, x_i is the difference of the source signal and the output signal times the value of the feedback factor evaluated at the appropriate frequency

$$x_i = X_s e^{j\omega t} - a_1'(j\omega) X_s e^{j\omega t} F(j\omega) - a_2'(j\omega) X_s^2 e^{j2\omega t} F(j2\omega) - \\ - a_3'(j\omega) X_s^3 e^{j3\omega t} F(j3\omega) \tag{7.91}$$

then it is processed by the nonlinear block whose output is

$$x_o = X_s [1 - F(j\omega) a_1'(j\omega)] e^{j\omega t} a_1(j\omega) - \\ - F(j2\omega) a_2'(j\omega) X_s^2 e^{j2\omega t} a_1(j2\omega) - F(j3\omega) a_3'(j\omega) X_s^3 e^{j3\omega t} a_1(j3\omega) + \\ + X_s^2 [1 - F(j\omega) a_1'(j\omega)]^2 e^{j2\omega t} a_2(j\omega) + \\ + X_s^3 [1 - F(j\omega) a_1'(j\omega)]^3 e^{j3\omega t} a_3(j\omega) \tag{7.92}$$

After substituting (7.90) in (7.92) and equating the terms with the same frequency component in the exponential factor, we get

$$a'_1 (j\omega) = [1 - F(j\omega) a'_1 (j\omega)] a_1(j\omega) \tag{7.93}$$

$$a'_2 (j\omega) = -F(j2\omega) a'_2 (j\omega) a_1(j2\omega) + [1 - F(j\omega) a'_1 (j\omega)]^2 a_2(j\omega) \tag{7.94}$$

$$a'_3 (j\omega) = -F(j\omega) a'_3 (j\omega) a_1(j3\omega) + [1 - F(j\omega) a'_1 (j\omega)]^3 a_3(j\omega) \tag{7.95}$$

Solving the above system for $a_1'(j\omega)$, $a_2'(j\omega)$, and $a_3'(j\omega)$ yields the same results as in (7.47), (7.48) and (7.49a).