if

$$\left\langle x, -\left[\frac{\partial f}{\partial x}\right]^T \operatorname{grad}_x T(x) \right\rangle = 0.$$

Proof: Consider the p.d.e.

$$\phi(q, x) = \langle q, f(x) \rangle = 0, \qquad q = \operatorname{grad}_x T(x). \tag{11}$$

Let

$$h(z) = \langle q, x \rangle.$$

Then

$$\begin{aligned} [h, \phi] &= \left\langle x, -\left[\frac{\partial f}{\partial x}\right]^T q \right\rangle + \langle q, f(x) \rangle \\ &= \left\langle x, -\left[\frac{\partial f}{\partial x}\right]^T \operatorname{grad}_x T(x) \right\rangle = 0. \end{aligned} \tag{20}$$

Hence, h and ϕ are in involution and, according to Theorem 3, h is constant along solutions of (19). Again

$$S_{x_i}^T = \frac{h(z)}{T(x)}$$

will also be constant.

Acknowledgment

The suthor would like to thank Profs. L. V. Boffi, D. Heim, and R. G. Stefanek for their kind interest.

Manoel Sobral, Jr.
Engineering Division
University of Michigan
Dearborn, Mich. 48128

References

[1] J. D. Schoeffler, "The synthesis of minimum sensitivity networks," *IEEE Trans. Circuit Theory*, vol. CT-11, pp. 271–276, June 1964.
[2] D. A. Calahan, "Computer generation of equivalent networks," *1964 IEEE Internatl. Conv. Rec.*, pt. 1, pp. 330–337.
[3] R. W. Newcomb, "The noncompleteness of continuously equivalent networks," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-13, pp. 207–208, June 1966.
[4] M. L. Blostein, "Generation of minimum sensitivity networks," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-14, pp. 87–88, March 1967.
[5] J. V. Leeds, Jr., and G. I. Ugron, "Simplified multiple parameter sensitivity calculation and continuously equivalent networks," *IEEE Trans. Circuit Theory*, vol. CT-14, pp. 188–191, June 1967.
[6] E. S. Kuh and C. G. Lau, "Sensitivity invariants of continuously equivalent networks," *IEEE Trans. Circuit Theory*, vol. CT-15, pp. 175–177, September 1968.
[7] For instance, C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, vol. 1. San Francisco: Holden Day, 1965.
[8] F. D. Murnaghan, *Differential Equation* (in Portuguese). Brazil: Centro Técnico de Aeronautica, S. José dos Campos, S. P., 1955.

Network Transfer Functions Using the Concept of Frequency-Dependent Negative Resistance

A positive immittance converter-type network (PIC) can be used to realize a frequency-dependent driving-point resistance (FDNR) and it has been suggested that the voltage transfer function of a passive LCR network can be realized by making use of the concept of a FDNR element [1]. The purpose of this correspondence is to prove that a general passive LCR network can be transformed to a topologically similar network, containing resistors, capacitors, and FDNR elements, which has the same voltage transfer function as the original LCR network. Furthermore, it is shown that the FDNR networks are particularly suitable for the realization of high-order low-pass ladder filters.

A one-port FDNR element is defined by its admittance, given by

Manuscript received November 7, 1968.

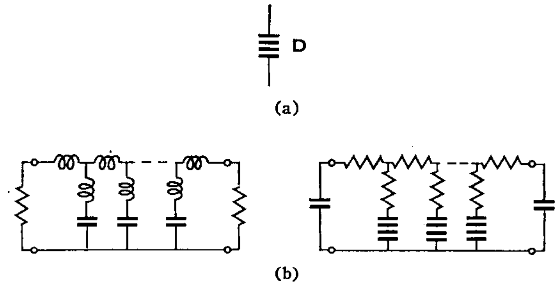

Fig. 1. (a) The FDNR element notation. (b) Equivalent LCR and DCR elliptic ladder filter networks.

$$y(s) = s^2 D \tag{1}$$

where D is a positive, real constant. The parameter D will be shown adjacent to the symbol given in Fig. 1(a) in order to denote the FDNR element. We define a DCR network as a network that can be considered as consisting of FDNR elements, resistors, and capacitors. The time description of the behavior of the FDNR element, in terms of the element current i and voltage v, is

$$i = D \frac{d^2 v}{dt^2}. \tag{2}$$

Now consider the transformation from a passive LCR network to an equivalent DCR network, where this DCR network is obtained by scaling the admittance levels of the LCR elements by the parameter s. Thus, if the admittance of the general LCR network branch is given by

$$Y_{LCR}(s) = G + sC + \frac{1}{sL} \tag{3}$$

where G, C, and L are the branch conductance, capacitance, and inductance, respectively, then the admittance of the general DCR network branch is defined as

$$Y_{DCR}(s) = sY_{LCR}(s) = sG + s^2C + \frac{1}{L} \tag{4}$$

where the equivalent DCR network branch will contain a capacitance G, resistance L, and FDNR element C.

Now if the LCR network is considered as a two-port with transmission matrix parameters A, B, C, and D, then it follows that, since A and D are immittance ratios, the equivalent DCR network will have identical A and D parameters to the LCR network. Also, since B and D have the dimensions of impedance and admittance, respectively, the admittance scaling implied by (4) will modify these parameters to B/s and Cs, respectively. Thus, it is concluded that the voltage transfer function A and current transfer function B are not altered as a result of the transformation from a LCR network to the equivalent DCR network.

Low-Pass Ladder Filter Realizations Using FDNR Elements

The realization of low-pass ladder filter voltage transfer functions is a useful practical application of the above transformation. Consider the general elliptic low-pass LCR prototype filter and the equivalent DCR filter as shown in Fig. 1(b). In this example, the DCR filter has capacitive terminations and the D elements each share a common terminal at ground potential. The FDNR realizations that have so far been suggested are most easily designed with one terminal grounded and therefore lend themselves to the realization of low-pass ladder DCR filters.

The DCR prototype network may be denormalized to a particular frequency ω_0 and a specific impedance level as follows:

$$C = \frac{C_n}{\omega_0 K}; \qquad R = R_n K; \qquad D = \frac{D_n}{\omega_0^2 K}$$

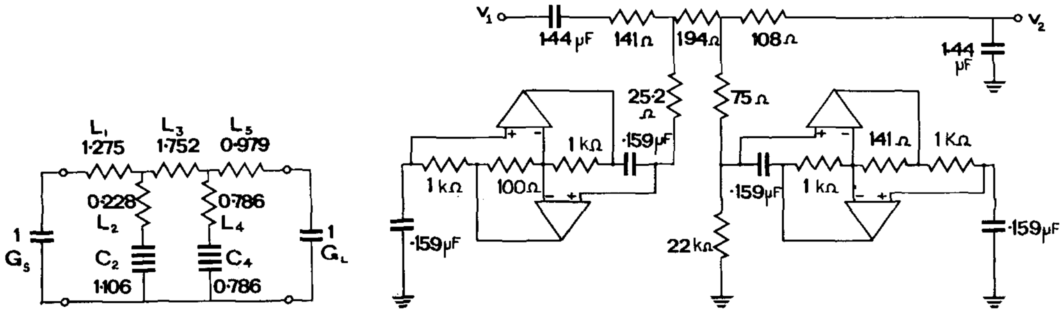

Fig. 2. (a) A prototype *DCR* elliptic ladder filter network. (b) A practical circuit realization of the elliptic ladder filter network.

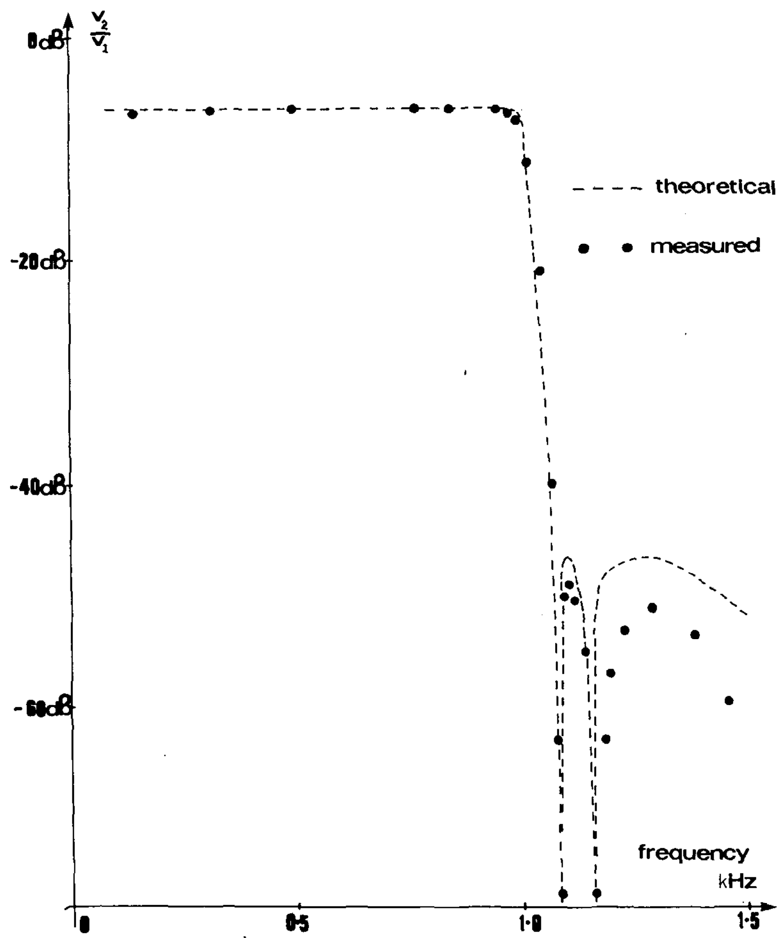

Fig. 3. Measured and ideal voltage transfer function of the elliptic ladder filter.

where the subscript n indicates the element value before denormalization and the constant K denormalizes to the required impedance levels. For example, the *DCR* filter given in Fig. 2(a) is denormalized so that

$$f_0 = \frac{\omega_0}{2\pi} = 1 \text{ kHz}$$

and

$$\frac{1}{\omega_0^2 D_2} = -100 \ \Omega$$

giving

$$K = 1.106 \times 100 = 110.6.$$

Thus, the denormalized element values, calculated from Fig. 2(a) are as follows:

$$L_1 = 141 \ \Omega; \quad L_2 = 25.2 \ \Omega; \quad L_3 = 194 \ \Omega;$$
$$L_4 = 75 \ \Omega \quad L_5 = 108 \ \Omega;$$
$$G_s = 1.44 \ \mu\text{F}; \quad G_L = 1.44 \ \mu\text{F};$$
$$D_2 = \left(\frac{1}{2\pi}\right)^2 \times 10^{-8}; \quad D_4 = \left(\frac{1}{2\pi}\right)^2 \times 0.71 \times 10^{-8}.$$

The complete *DCR* filter network is given in Fig. 2(b), where the D elements are realized using the network that is analysed in [2]. The 22-kΩ resistor in Fig. 2(b) is used to provide input bias current to the μ A702C amplifiers and will appear in the corresponding *LCR* filter as a large nonideal shunt inductor. This will cause the low-frequency response of the filter to fall off to zero.

The measured and ideal voltage transfer functions are given in Fig. 3, where it may be seen that the measured transfer function is in good agreement with theory. It has been shown that the particular PII/PIC network used in this realization is particularly useful at high frequencies because it exhibits a large Q factor. Further experimental work has demonstrated that very high-order ladder filters can be constructed using this technique with cutoff frequencies up to 250 kHz with μA702C amplifiers. It should be noted that a general high-order elliptic low-pass LC filter, consisting of N floating inductors, requires $(N - 1)$ operational amplifiers if it is realized using this method. Furthermore, comments made elsewhere [3], [4] concerning network sensitivity also apply to this type of realization, and the overall sensitivity is comparable to other methods employing PII's and PIC's [5]–[8].

To summarize, a class of DCR networks has been suggested, which is obtained by direct transformation from LCR networks, and it has been shown that the transformation is particularly useful for realizing active ladder filters in which the LCR version contains floating inductors. Work is continuing with the applications of impedance conversion networks to the realization of active filters, oscillators, modulators, and mixers. The results of this work will be published as they become available.

L. T. Bruton
Dept. of Elec. Engrg.
The Merz Labs.
University of Newcastle upon Tyne
Newcastle upon Tyne, England

References

[1] L. T. Bruton, "Frequency selectivity using positive impedance converter-type networks," *Proc. IEEE* (Letters), vol. 56, pp. 1378–1379, August 1968.
[2] ——, "Non-ideal performance of a class of positive immittance converters," *Proc. IEEE* (Letters), February 1970, (to be published).
[3] H. J. Orchard, "Inductorless filters," *Electron. Letters*, vol. 2, pp. 224–225, June 1966.
[4] J. Gorski-Popiel, "RC-active synthesis using positive-immittance convertors," *Electron. Letters*, vol. 3, pp. 381–382, August 1967.
[5] B. A. Shenoi, "Practical realization of a gyrator circuit and RC-gyrator filters," *IEEE Trans. Circuit Theory*, vol. CT-12, pp. 374–380, September 1965.
[6] W. H. Holmes, S. Gruetzmann, and W. E. Heinlein, "Direct-coupled gyrators with floating ports," *Electron. Letters*, vol. 3, pp. 45–46, February 1967.
[7] D. F. Sheahan, "Gyrator flotation circuit," *Electron. Letters*, vol. 3, pp. 39–40, January 1967.
[8] D. G. Lampard and G. A. Rigby, "The application of a positive immittance inverter to the design of integrated selective filters," *Proc. IEEE* (Letters), vol. 55, pp. 1101–1102, June 1967.

Chebyshev Rational Functions and Ladder RC Networks

In working with optimum RC phase-shift problems [1], [2], the author has come across a class of functions that can be used in a new approach to the study of RC ladder networks. The aim of this correspondence is to introduce these functions, and then use these functions 1) to evaluate the zero positions of the $ABCD$ parameters of a geometric progression ladder RC network, and 2) to study the asymptotic behavior of these zero locations as the number of sections in the network increases.

I. The $ABCD$ Parameters of a Geometric Progression Ladder RC Network and Their Zero Positions

1) Consider the n-section RC ladder network of Fig. 1, the resistance and capacitance distributions of which are defined by

$$R_{i+1} = aR_i, \qquad i = 1, \cdots, n-1 \tag{1}$$

$$R_iC_i = T, \qquad i = 1, \cdots, n. \tag{2}$$

Manuscript received October 9, 1968; revised December 21, 1968.

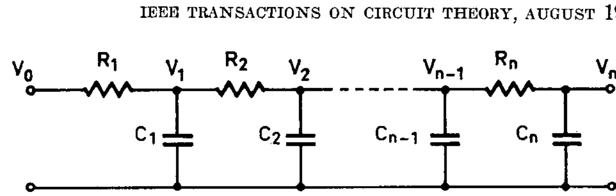

Fig. 1. The geometric progression RC ladder network.

Application of Kirchoff's current law at node i yields the following second-order linear difference equation

$$\frac{V_{i+1}}{\sqrt{a}} - V_i\sqrt{a}\left(1 + \frac{1}{a} + PT\right) + V_{i+1}\sqrt{a} = 0, \qquad i = 1, \cdots, n-1 \tag{3}$$

the solution of which is given by

$$V_i = a^{i/2}[M \cosh (n-1)\phi_1 + N \sinh (n-i)\phi_1], \qquad i = 0, 1, \cdots, n \tag{4}$$

where

$$\cosh \phi_1 = \frac{\sqrt{a}}{2}\left(1 + \frac{1}{a} + PT\right), \tag{5}$$

M, N are two constant multipliers independent of the index i, and P is the complex frequency.

By applying various terminating conditions to (4), the $ABCD$ parameters of the general progression network can be derived in the following form [3], [4].

$$A(P) = a^{-n/2}\cdot\frac{\sinh [n+1]\phi_1}{\sinh \phi_1} - a^{-(n+1)/2}\cdot\frac{\sinh [n\phi_1]}{\sinh \phi_1} \tag{6}$$

$$B(P) = R_1 a^{(n-1)/2}\cdot\frac{\sinh [n\phi_1]}{\sinh \phi_1} \tag{7}$$

$$C(P) = \frac{1}{R_1}\left\{a^{-n/2}\cdot\frac{\sinh [(n+1)\phi_1]}{\sinh \phi_1} - a^{-(n-1)/2}\cdot\frac{\sinh [n_1]}{\sinh \phi_1} - a^{-(n+1)/2}\cdot\frac{\sinh [n\phi_1]}{\sinh \phi_1} + a^{-n/2}\cdot\frac{\sinh [(n-1)\phi_1]}{\sinh \phi_1}\right\} \tag{8}$$

$$D(P) = a^{(n-1)/2}\frac{\sinh [n\phi_1]}{\sinh \phi_1} - a^{n/2}\frac{\sinh [(n-1)\phi_1]}{\sinh \phi_1}. \tag{9}$$

In analyzing the transient response of the ladder network, it is often desirable to know the exact zero positions of $A(P)$. Since the expression (6) for $A(P)$ comprises two separate terms, both of which are oscillatory in nature, these zero positions are numerically difficult to evaluate. In the following, a more tractable expression will be sought for the determination of these zero positions.

2) Consider the following rational functions in P.

$$\cosh \phi_1 = \frac{P + \frac{1}{2}(t_\alpha + t_\beta)}{\frac{1}{2}(t_\alpha - t_\beta)} \tag{10}$$

$$\cosh \phi_2 = \frac{(t_\alpha + t_\beta)P + 2t_\alpha t_\beta}{(t_\beta - t_\alpha)P}. \tag{11}$$

These two functions are akin to the hyperbolic cosine functions used in filter network synthesis [5], [6]. In the same way, it can be shown that $\cosh \{j\phi_1 + k\phi_2\}$ is a rational function in P provided j and k are integers, and that the function oscillates between $+1$ and -1 $(j + k)$ times, with a k-order pole at the origin. The "passband"[1]

[1] In accordance with the terminology used in filter synthesis theory.